

Numerical approximation of stochastic evolution equations: Convergence in scale of Hilbert spaces[☆]

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Abstract

The paper is devoted to the numerical approximation of a very general stochastic nonlinear evolution equation in a separable Hilbert space H . Examples of such equations which fall into our framework include the GOY and sabra shell models and a nonlinear heat equation. The space-time numerical scheme is defined in terms of a Galerkin approximation in space and by a semi implicit finite difference approximation in time (Rothe approximation). We prove the convergence in probability of our scheme. This is shown by means of an estimate of the error on a localized set of arbitrary large probability. Let us mention that our error estimate is shown to hold in a more regular space V_β with $\beta \in [0, \frac{1}{4})$ and that the speed of convergence depends on this parameter β . Also, an explicit rate is given as a consequence.

Keywords: Goy and sabra shell model, nonlinear heat equation, Galerkin approximation, time discretization, fully implicit scheme, Rothe scheme, semi-implicit scheme, convergence in probability

1. Introduction

Throughout this paper we fix a complete filtered probability space $\mathfrak{U} = (\Omega, \mathcal{F}_t, \mathbb{F}, \mathbb{P})$ with the filtration $\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$ satisfying the usual conditions. Throughout this paper we fix a separable Hilbert space H equipped with a scalar product (\cdot, \cdot) with the associated norm $|\cdot|$. We fix also another separable Hilbert space \mathcal{H} .

In this paper, we analyze numerical approximations of the following stochastic evolution equation

$$\begin{aligned} d\mathbf{u} &= -[A\mathbf{u} + B(\mathbf{u}, \mathbf{u})]dt + G(\mathbf{u})dW(t), \quad t \in [0, T], \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{aligned} \tag{1.1}$$

where A is a self-adjoint positive operators on H , B , and G are nonlinear maps satisfying several technical assumptions to be specified later and $W = \{W(t); 0 \leq t \leq T\}$ a \mathcal{H} -valued Wiener process.

The abstract equation (1.1) can describe several problems from different fields including mathematical finance, electromagnetism, and fluid dynamic. Stochastic models are well fitted to describe small fluctuations or perturbations which arises in nature. The stochastic Navier–Stokes equations is used to describe fully turbulent fluids via the celebrated K41 theory [1], see also [2, 3] and reference therein. The reader interested to other models involving the introduction of a stochastic term are invited to consult [4].

Due to the nonlinearity B , it is very difficult to derive an exact solution (or closed-form solution) stochastic evolution equations such as (1.1), a numerical resolution is then inevitable. The field of

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numerical analysis for SPDEs or stochastic evolution equations is still young and the numerical approximation of these equations is not trivial when the nonlinear term B is not Lipschitz which is the case for stochastic Navier–Stokes equations type. Recent works involving a bilinear term have been done including [5, 6]. In [6] a weak martingale to the incompressible Navier-Stokes equations with Gaussian multiplicative noise is constructed from a convergent finite element based on space-time discretization, and the authors of [5] proved the convergence rates of an explicit and an implicit numerical schemes by means of a Gronwall argument. The main issue when the term B is not Lipschitz lies on its interplay with the stochastic forcing, which prevents a Gronwall argument in the context of expectations. This issue is for example solved in [7, 8] by the introduction of a weight, which when carefully chosen will contribute to remove unwanted terms and allow to use Gronwall lemma. In [5], the authors use different approach by computing the error estimates on a sample subset $\Omega_k \subset \Omega$ with large probability. In particular, the set Ω_k is carefully chosen so that the random variables $\|\nabla \mathbf{u}^\ell\|_{L^2}$ are bounded as long as the events are taken in Ω_k , and $\lim_{k \searrow 0} \mathbb{P}(\Omega \setminus \Omega_k) = 0$. Then, standard machinery using the Gronwall lemma are performed.

In this paper, we discretize (1.1) using a coupled Galerkin method and (semi-)implicit Euler scheme and show convergence with rates. Regardless of the approach, this work is similar to [5] but in a smaller space, i.e. we use a Gronwall argument and utilize Hölder continuity in time of \mathbf{u} in some Hilbert space to arrive at convergence in probability in V_β with rate $\theta_0 \in (0, 1/4 - \beta)$ where $\beta \in [0, \frac{1}{4})$ is arbitrary. In contrast to the nonlinear term of Navier–Stokes equations with periodic boundary condition treated in [5], our nonlinear term B is allowed to non-bilinear and it does not satisfy the property $\langle B(\mathbf{u}, \mathbf{u}), A\mathbf{u} \rangle = 0$ which plays a crucial role in the analysis in [5]. Examples of semilinear equations which fall into our framework include the GOY and sabra shell models. These models are created in view of simulating turbulence, but it seems that our work is the first one rigorously addressing their numerical analysis. Our result also confirm that, in term of numerical analysis, the shell models behave far better than the Navier-Stokes. Another example such as a nonlinear heat equation are described in Section 5. We should also note that we also give a new and simple proof of the existence of solutions to stochastic shell models driven by Gaussian multiplicative noise.

This paper is organized as follows: In Section 2, we introduce the necessary notations and the standing assumptions that will be used in the present work. In Section 3, we present our numerical scheme and also discuss the stability and existence of solution at each time step. The convergence of the proposed method in the mean-square sense is presented in Section 4. In Section 5 we present the stochastic shell models for turbulence and stochastic nonlinear heat equation as a motivating examples.

2. Notations, assumptions, preliminary results and the main theorem

In this section we introduce the necessary notations and the standing assumptions that will be used in the present work. We will also introduce our numerical scheme and state our main result.

2.1. Assumptions and notations

Throughout this paper we fix a separable Hilbert space H with norm $|\cdot|$ and a fixed orthonormal basis $\{\psi_n; n \in \mathbb{N}\}$. We assume that we are given a linear operator $A : D(A) \subset H \rightarrow H$ which is a self-adjoint and positive operator such that the fixed basis $\{\psi_n; n \in \mathbb{N}\}$ satisfies

$$\{\psi_n; n \in \mathbb{N}\} \subset D(A), \quad A\psi_n = \lambda_n \psi_n,$$

for an increasing sequence of positive numbers $\{\lambda_n; n \in \mathbb{N}\}$ with $\lambda_n \rightarrow \infty$ as $n \nearrow \infty$. It is clear that $-A$ is the infinitesimal generator of an analytic semigroup $e^{-tA}, t \geq 0$, on H . For any $\alpha \in \mathbb{R}$ the domain of A^α denoted by $V_\alpha = D(A^\alpha)$ is a separable Hilbert space when equipped with the scalar product

$$((\mathbf{u}, \mathbf{v}))_\alpha = \sum_{k=1}^{\infty} \lambda_k^{2\alpha} \mathbf{u}_k \mathbf{v}_k, \text{ for } \mathbf{u}, \mathbf{v} \in V_\alpha. \quad (2.1)$$

The norm associated to this scalar product will be denoted by $\|\mathbf{u}\|_\alpha$, $\mathbf{u} \in V_\alpha$. In what follows we set $V := D(A^{\frac{1}{2}})$.

Next, we consider a nonlinear map $B(\cdot, \cdot) : V \times V \rightarrow V^*$ satisfying the following set of assumptions, where hereafter Y^* denotes the dual of the Banach space Y .

(B1) There exists a constant $C_0 > 0$ such that for any $\theta \in [0, \frac{1}{2})$ and $\gamma \in (0, \frac{1}{2})$ satisfying $\theta + \gamma \in (0, \frac{1}{2})$, we have

$$\|B(\mathbf{u}, \mathbf{v}) - B(\mathbf{x}, \mathbf{y})\|_{-\theta} \leq \begin{cases} C_0 \|\mathbf{u} - \mathbf{x}\|_{\frac{1}{2} - (\theta + \gamma)} (\|\mathbf{v}\|_\gamma + C\|\mathbf{y}\|_\gamma) + \|\mathbf{v} - \mathbf{y}\|_\gamma (\|\mathbf{u}\|_{\frac{1}{2} - (\theta + \gamma)} + \|\mathbf{x}\|_{\frac{1}{2} - (\theta + \gamma)}) \\ \text{for any } \mathbf{u}, \mathbf{x} \in V_{\frac{1}{2} - (\theta + \gamma)} \text{ and } \mathbf{v}, \mathbf{y} \in V_\gamma, \\ C_0 (\|\mathbf{u}\|_\gamma + \|\mathbf{x}\|_\gamma) \|\mathbf{v} - \mathbf{y}\|_{\frac{1}{2} - (\theta + \gamma)} + C\|\mathbf{u} - \mathbf{x}\|_\gamma (\|\mathbf{v}\|_{\frac{1}{2} - (\theta + \gamma)} + \|\mathbf{y}\|_{\frac{1}{2} - (\theta + \gamma)}) \\ \text{for any } \mathbf{v}, \mathbf{y} \in V_{\frac{1}{2} - (\theta + \gamma)}, \text{ and } \mathbf{u}, \mathbf{x} \in V_\gamma. \end{cases} \quad (2.2)$$

If $\theta = \frac{1}{2}$, then we assume that there exists a constant $\gamma > 0$ such that (2.2) holds with $\frac{1}{2} - (\theta + \gamma)$ replaced by $\frac{1}{2} - \gamma$.

In addition to the above, we also assume that for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$|B(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}\| \|\mathbf{v}\|_{\frac{1}{2} + \varepsilon}, \text{ for any } \mathbf{u} \in H, \mathbf{v} \in V_{\frac{1}{2} + \varepsilon}. \quad (2.3)$$

(B2) There exists a positive number \varkappa such that for any $\mathbf{u}, \mathbf{v} \in V$

$$\langle A\mathbf{v} + B(\mathbf{u}, \mathbf{v}), \mathbf{v} \rangle \geq \varkappa \|\mathbf{v}\|_{\frac{1}{2}}^2. \quad (2.4)$$

Without loss of generality we will assume that $\varkappa = 1$ for the remaining part of the paper.

(B3) We assume that for any $\mathbf{u} \in H$ we have

$$B(0, \mathbf{u}) = B(\mathbf{u}, 0) = 0. \quad (2.5)$$

Note that Assumptions (B1) and (B3) imply

(B1)' There exists a constant $C_0 > 0$ such that for any numbers $\theta \in [0, \frac{1}{2})$ and $\gamma \in (0, \frac{1}{2})$ satisfying $\theta + \gamma \in (0, \frac{1}{2})$, we have

$$\|B(\mathbf{u}, \mathbf{v})\|_{-\theta} \leq C_0 \begin{cases} \|\mathbf{u}\|_{\frac{1}{2} - (\theta + \gamma)} \|\mathbf{v}\|_\gamma & \text{for any } \mathbf{u} \in V_{\frac{1}{2} - (\theta + \gamma)} \text{ and } \mathbf{v} \in V_\gamma, \\ \|\mathbf{u}\|_\gamma \|\mathbf{v}\|_{\frac{1}{2} - (\theta + \gamma)} & \text{for any } \mathbf{v} \in V_{\frac{1}{2} - (\theta + \gamma)}, \text{ and } \mathbf{u} \in V_\gamma. \end{cases} \quad (2.6)$$

If $\theta = \frac{1}{2}$, then we assume that there exists a real number $\gamma > 0$ such that (2.6) holds with $\frac{1}{2} - (\theta + \gamma)$ replaced by $\frac{1}{2} - \gamma$.

Now, let $\mathcal{U} := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where the filtration $\mathbb{F} = \{\mathcal{F}_t; t \in [0, T]\}$ satisfies the usual condition. Let $\{\mathbf{w}_j; j \in \mathbb{N}\}$ be a sequence of mutually independent and identically distributed standard Brownian motions on \mathcal{U} . Let \mathcal{H} be separable Hilbert space and $\mathcal{L}_1(\mathcal{H})$ be the space of all trace class operators on \mathcal{H} . Recall that if $Q \in \mathcal{L}_1(\mathcal{H})$ is a symmetric, positive operator and $\{\varphi_j; j \in \mathbb{N}\}$ is an orthonormal basis of \mathcal{H} consisting of eigenvectors of Q , then the series

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j} \mathbf{w}_j(t) \varphi_j, \quad t \in [0, T],$$

where $\{q_j; j \in \mathbb{N}\}$ are the eigenvalues of Q , converges in $L^2(\Omega; C([0, T]; \mathcal{H}))$ and it defines an \mathcal{H} -valued Wiener process with covariance operator Q . Furthermore, for any positive integer $\ell > 0$ there exists a constant $C_\ell > 0$ such that

$$\mathbb{E} \|W(t) - W(s)\|_{\mathcal{H}}^{2\ell} \leq C_\ell |t - s|^\ell (\text{Tr } Q)^\ell, \quad (2.7)$$

for any $t, s \geq 0$ with $t \neq 0$. Before proceeding further we recall few facts about stochastic integral. Let K be a separable Hilbert space, $\mathcal{L}(\mathcal{H}, K)$ be the space of all bounded linear K -valued operators defined on \mathcal{H} , $\mathcal{M}_T^2(K) := \mathcal{M}^2(\Omega \times [0, T]; K)$ be the space of all equivalence classes of \mathbb{F} -progressively measurable processes $\Psi : \Omega \times [0, T] \rightarrow K$ satisfying

$$\mathbb{E} \int_0^T \|\Psi(s)\|_K^2 ds < \infty.$$

If $Q \in \mathcal{L}_1(\mathcal{H})$ is a symmetric, positive and trace class operator then $Q^{\frac{1}{2}} \in \mathcal{L}_2(\mathcal{H})$ and for any $\Psi \in \mathcal{L}(\mathcal{H}, K)$ we have $\Psi \circ Q^{\frac{1}{2}} \in \mathcal{L}_2(\mathcal{H}, K)$, where $\mathcal{L}_2(\mathcal{H}, K)$ (with $\mathcal{L}_2(\mathcal{H}) := \mathcal{L}_2(\mathcal{H}, \mathcal{H})$) is the Hilbert space of all operators $\Psi \in \mathcal{L}(\mathcal{H}, K)$ satisfying

$$\|\Psi\|_{\mathcal{L}_2(\mathcal{H}, K)}^2 = \sum_{j=1}^{\infty} \|\Psi \varphi_j\|_K^2 < \infty.$$

Furthermore, from the theory of stochastic integration on infinite dimensional Hilbert space, for any $\Psi \in M_T^2(\mathcal{L}(\mathcal{H}, K))$ the process M defined by

$$M(t) = \int_0^t \Psi(s) dW(s), t \in [0, T],$$

is a K -valued martingale. Moreover, we have the following Itô isometry

$$\mathbb{E} \left(\left\| \int_0^t \Psi(s) dW(s) \right\|_K^2 \right) = \mathbb{E} \left(\int_0^t \|\Psi(s) Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{H}, K)}^2 ds \right), \forall t \in [0, T], \quad (2.8)$$

and the Burkholder-Davis-Gundy inequality

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} \left\| \int_0^s \Psi(s) dW(s) \right\|^q \right) \leq C_q \mathbb{E} \left(\int_0^t \|\Psi(s) Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{H}, K)}^2 ds \right)^{\frac{q}{2}}, \forall t \in [0, T], \forall q \in (1, \infty). \quad (2.9)$$

Now, we impose the following set of conditions on the nonlinear term $G(\cdot)$ and the Wiener process W .

- (N) Let \mathcal{H} be a separable Hilbert space. We assume that the driving noise W is a \mathcal{H} -valued Wiener process with a positive and symmetric covariance operator $Q \in \mathcal{L}_1(\mathcal{H})$.
- (G) We assume that for the nonlinear function G maps H to $\mathcal{L}(\mathcal{H}, V_{\frac{1}{4}})$ and that there exists a constant $C_1 > 0$ such that for any $\mathbf{u} \in H, \mathbf{v} \in H$ we have

$$\|G(\mathbf{u}) - G(\mathbf{v})\|_{\mathcal{L}(\mathcal{H}, V_{\frac{1}{4}})} \leq C_1 |\mathbf{u} - \mathbf{v}|.$$

Remark 2.1.

- (a) Note that the above assumption implies that $G : H \rightarrow \mathcal{L}(\mathcal{H}, H)$ is globally Lipschitz and of at most linear growth, i.e, there exists a constant $C_2 > 0$ such that

$$\begin{aligned} \|G(\mathbf{u}) - G(\mathbf{v})\|_{\mathcal{L}(\mathcal{H}, H)} &\leq C_2 |\mathbf{u} - \mathbf{v}|, \\ |G(\mathbf{u})| &\leq C_2 (1 + |\mathbf{u}|), \end{aligned}$$

for any $\mathbf{u}, \mathbf{v} \in H$.

- (b) There also exists a number $C_3 > 0$ such that

$$\begin{aligned} \|G(\mathbf{u}) - G(\mathbf{v})\|_{\mathcal{L}(\mathcal{H}, V_{\frac{1}{4}})} &\leq C_3 \|\mathbf{u} - \mathbf{v}\|_{\frac{1}{4}}, \\ \|G(\mathbf{u})\|_{\mathcal{L}(\mathcal{H}, V_{\frac{1}{4}})} &\leq C_3 (1 + \|\mathbf{u}\|_{\frac{1}{4}}), \end{aligned}$$

for any $\mathbf{u}, \mathbf{v} \in V_{\frac{1}{4}}$.

- (c) Owing to item (a) of the present remark, if $\mathbf{u} \in M_T^2(H)$, then $G(\mathbf{u}) \in M_T^2(\mathcal{L}(\mathcal{H}, H))$ and the stochastic integral $\int_0^t G(\mathbf{u}(s)) dW(s)$ is a well defined H -valued martingale.

2.2. Preliminary results

In this subsection we recall and derive some preliminary results that we will be using in the remaining part of the paper. To this end, we first define the notion of solution of (1.1).

Definition 2.2. An \mathbb{F} -adapted process \mathbf{u} is called a weak solution of Equation (1.1) (in the sense of duality) if the following conditions are satisfied

- (i) $\mathbf{u} \in L^2(0, T; V) \cap C([0, T]; H)$ \mathbb{P} -a.s.,
- (ii) the following equality holds for every $t \in [0, T]$ and \mathbb{P} -a.s.,

$$(\mathbf{u}(t), \phi) = (\mathbf{u}_0, \phi) - \int_0^t (\langle A\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s), \mathbf{u}(s)), \phi \rangle) ds + \int_0^t \langle G(\mathbf{u}(s)), \phi \rangle dW(s), \quad (2.10)$$

for any $\phi \in V$.

Next, we recall the following result.

Proposition 2.3. If the assumptions (B1) to (B3) hold and (G) is satisfied with $V_{\frac{1}{4}}$ replaced by H and $\mathbf{u}_0 \in L^2(\Omega, H)$, then the problem (1.1) has a unique global mild, which is also a weak, solution \mathbf{u} . Let $T > 0$ be a real number. moreover, if $\mathbf{u}_0 \in L^{2p}(\Omega, H)$ for any real number $p \in [2, 8]$, then there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_{t \in [0, T]} |\mathbf{u}(t)|^{2p} + \mathbb{E} \int_0^T |\mathbf{u}(s)|^{2p-2} |A^{\frac{1}{2}} \mathbf{u}(s)|^2 ds \leq C(1 + \mathbb{E} |\mathbf{u}_0|^{2p}), \quad (2.11)$$

and

$$\mathbb{E} \left(\int_0^T |A^{\frac{1}{2}} \mathbf{u}(s)|^2 ds \right)^p \leq C(1 + \mathbb{E} |\mathbf{u}_0|^{2p}). \quad (2.12)$$

If, in addition, Assumption (G) is satisfied and $\mathbf{u}_0 \in L^p(\Omega, V_{\frac{1}{4}})$, with $p \in [2, 8]$, $T > 0$, then there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_{t \in [0, T]} \|\mathbf{u}(t)\|_{\frac{1}{4}}^p + \mathbb{E} \left(\int_0^T \|\mathbf{u}(s)\|_{\frac{3}{4}} ds \right)^p \leq C(1 + \mathbb{E} \|\mathbf{u}_0\|_{\frac{1}{4}}^p + (\mathbb{E} |\mathbf{u}_0|^{2p})^2). \quad (2.13)$$

Proof. Let us first prove the existence of a local mild solution. For this purpose, we study the properties of B in order to apply a contraction principle as in [9, Theorem 3.15]. Let $B(\cdot)$ be the mapping defined by $B(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$ for any $\mathbf{x} \in V_\beta$. Let $\beta \in (0, \frac{1}{2})$. Using Assumptions (B1) with $\theta = \frac{1}{2} - \beta$, $\gamma = \beta$, we derive that

$$\|B(\mathbf{x}) - B(\mathbf{y})\|_{\beta - \frac{1}{2}} \leq C_0 |\mathbf{x} - \mathbf{y}| (\|\mathbf{x}\|_\beta + \|\mathbf{y}\|_\beta) + C \|\mathbf{x} - \mathbf{y}\|_\beta (|\mathbf{x}| + |\mathbf{y}|), \quad (2.14)$$

for any $\mathbf{x}, \mathbf{y} \in V_\beta$. Since, by [10, Theorem 1.18.10, pp 141], V_β coincide with the complex interpolation $[H, D(A^{\frac{1}{2}})]_{2\beta}$, we infer from the interpolation inequality [10, Theorem 1.9.3, pp 59] and (2.14) that

$$\|B(\mathbf{x}) - B(\mathbf{y})\|_{\beta - \frac{1}{2}} \leq C_0 |\mathbf{x} - \mathbf{y}| (|\mathbf{x}|^{1-2\beta} \|\mathbf{x}\|_{\frac{1}{2}}^{2\beta} + |\mathbf{y}|^{1-2\beta} \|\mathbf{y}\|_{\frac{1}{2}}^{2\beta}) + C \|\mathbf{x} - \mathbf{y}\|_{\frac{1}{2}}^{2\beta} |\mathbf{x} - \mathbf{y}|^{1-2\beta} (|\mathbf{x}| + |\mathbf{y}|), \quad (2.15)$$

for any $\mathbf{x}, \mathbf{y} \in V$. Now, we denote by X_T the Banach space $C([0, T]; H) \cap L^2(0, T; V)$ endowed with the norm

$$\|\mathbf{x}\|_{X_T} = \sup_{t \in [0, T]} |\mathbf{x}(t)| + \left(\int_0^T \|\mathbf{x}(t)\|_{\frac{1}{2}}^2 dt \right)^{\frac{1}{2}}.$$

We recall the following classical result (see [11, Theorem 3, pp 520])

$$\text{The linear map } \Lambda : L^2(0, T; V^*) \ni f \mapsto \mathbf{x}(\cdot) = \int_0^\cdot e^{-(\cdot-r)A} f(r) dr \in X_T \text{ is continuous.} \quad (2.16)$$

Thus, thanks to (2.15), (2.16) and Assumption (G) we can apply [9, Theorem 3.15] to infer the existence of a unique local mild solution \mathbf{u} with lifespan τ of (1.1) (we refer to [9, Definition 3.1] for the definition of local solution). Let $\{\tau_j; j \in \mathbb{N}\}$ be an increasing sequence of stopping times converging almost surely to the lifespan τ . Using the equivalence lemma in [12, Proposition 6.5] we can easily prove that the local mild solution is also a local weak solution satisfying (2.10) with t replaced by $t \wedge \tau_j, j \in \mathbb{N}$. Now, we can prove by arguing as in [13, Appendix A] or [8, Proof of Theorem 4.4] that the local solution \mathbf{u} satisfies (2.11) uniformly w.r.t. $j \in \mathbb{N}$. With this observation along with an argument similar to [9, Proof of Theorem 2.10] we conclude that (1.1) admits a global solution \mathbf{u} satisfying (2.11) and $\mathbf{u} \in X_T$ almost-surely. The proof of (2.12) is very similar to that of (2.13), hence below we will only show (2.13).

To prove (2.13) it suffices to treat the case $p \in \mathbb{N} \cap (2, 8]$, in particular $p = 8$. To start with we will formally apply the Itô formula to $\varphi(\mathbf{u}) = \|\mathbf{u}\|_{\frac{1}{4}}^2$. The following calculations can be rigorously justified by using the Galerkin truncation.

Applying the Itô formula to the functional $\varphi(\mathbf{u})(t) = \|\mathbf{u}(t)\|_{\frac{1}{4}}^2$

$$d\varphi(\mathbf{u}(t)) = \varphi'(\mathbf{u}(t))d\mathbf{u}(t) + \frac{1}{2} \text{Tr}(\varphi''(\mathbf{u}(t))G(\mathbf{u}(t))Q(G\mathbf{u}(t))^*)dt,$$

which along with the inequality $\frac{1}{2}\|\phi''(\mathbf{u})\| \leq 1$, where the norm is understood as the norm of a bilinear map, implies

$$\begin{aligned} d\|\mathbf{u}(t)\|_{\frac{1}{4}}^2 + [2\|\mathbf{u}(t)\|_{\frac{3}{4}}^2 + 2\langle A^{\frac{1}{2}}\mathbf{u}(t), B(\mathbf{u}(t), \mathbf{u}(t)) \rangle]dt &\leq 2\langle A^{\frac{1}{2}}\mathbf{u}(t), G(\mathbf{u}(t)) \rangle dW(t) \\ &+ C \text{Tr} Q \|G(\mathbf{u}(t))\|_{\mathcal{L}(\mathcal{H}, V_{\frac{1}{4}})}^2 dt. \end{aligned} \quad (2.17)$$

Since the embedding $V_{\frac{1}{2}+\alpha} \subset V_{2\alpha}$ is continuous for any $\alpha \in [0, \frac{1}{2}]$, we can use Assumptions (B1)' and the Cauchy inequality to infer that

$$\begin{aligned} \left| \int_0^T \langle A^{\frac{1}{2}}\mathbf{u}(s), B(\mathbf{u}(s), \mathbf{u}(s)) \rangle ds \right| &\leq C \int_0^T \|\mathbf{u}(s)\|_{\frac{1}{2}} \|B(\mathbf{u}(s), \mathbf{u}(s))\| ds, \\ &\leq \frac{1}{2} \int_0^T \|\mathbf{u}(s)\|_{\frac{3}{4}}^2 ds + C \int_0^T \|\mathbf{u}(s)\|_{\frac{1}{2}-\gamma}^2 \|\mathbf{u}(s)\|_{\gamma}^2 ds, \end{aligned}$$

for some $\gamma \in (0, \frac{1}{2})$. By complex interpolation inequality in [10, Theorem 1.9.3, pp 59] we have

$$\left| \int_0^T \langle A^{\frac{1}{2}}\mathbf{u}(s), B(\mathbf{u}(s), \mathbf{u}(s)) \rangle ds \right| \leq \frac{1}{2} \int_0^T \|\mathbf{u}(s)\|_{\frac{3}{4}}^2 ds + \int_0^T |\mathbf{u}(s)|^2 \|\mathbf{u}(s)\|_{\frac{1}{2}}^2 ds.$$

Plugging the latter inequality into (2.17), using the assumption on G we obtain

$$\begin{aligned} \|\mathbf{u}(t)\|_{\frac{1}{4}}^2 + \frac{3}{2} \int_0^t \|\mathbf{u}(s)\|_{\frac{3}{4}}^2 ds &\leq \|\mathbf{u}(0)\|_{\frac{1}{4}}^2 + C \sup_{s \in [0, T]} |\mathbf{u}(s)|^2 \int_0^t \|\mathbf{u}(s)\|_{\frac{1}{2}}^2 ds \\ &+ CT + C \int_0^t \|\mathbf{u}(s)\|_{\frac{1}{4}}^2 ds + 2 \left| \int_0^t \langle A^{\frac{1}{2}}\mathbf{u}(s), A^{\frac{1}{2}}G(\mathbf{u}(s)) \rangle dW(s) \right|. \end{aligned} \quad (2.18)$$

Taking the supremum over $t \in [0, T]$, then raising both sides of the resulting inequality to the power $p/2$, taking the mathematical expectation, using the Burkholder-Davis-Gundy inequality yield

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} \|\mathbf{u}(s)\|_{\frac{1}{4}}^p + 2\mathbb{E} \left(\int_0^t \|\mathbf{u}(s)\|_{\frac{3}{4}}^2 ds \right)^{p/2} &- \left(C\mathbb{E} \|\mathbf{u}(0)\|_{\frac{1}{4}}^p + CT + \mathbb{E} \int_0^t \|\mathbf{u}(s)\|_{\frac{1}{4}}^p ds \right) \\ &\leq C \left(\mathbb{E} \sup_{s \in [0, T]} |\mathbf{u}(s)|^{2p} \right)^{\frac{1}{2}} \left[\mathbb{E} \left(\int_0^T \|\mathbf{u}(s)\|_{\frac{1}{2}}^2 ds \right)^p \right]^{\frac{1}{2}} \\ &+ 2C\mathbb{E} \left(\int_0^t |A^{\frac{1}{2}}\mathbf{u}(s)|^2 \|G(\mathbf{u}(s))\|_{\mathcal{L}(\mathcal{H}, V_{\frac{1}{4}})}^2 ds \right)^{\frac{p}{4}}. \end{aligned} \quad (2.19)$$

Here we have used the fact that for any integer ℓ and n we can find a constant $C_{\ell,n}$ such that

$$\sum_{i=1}^n a_i^\ell \leq \left(\sum_{i=1}^n a_i \right)^\ell \leq C_{\ell,n} \sum_{i=1}^n a_i^\ell \quad (2.20)$$

for a sequence of non-negative numbers $\{a_i; i = 1, 2, \dots, n\}$.

Invoking (2.11), (2.12), the assumption on G we derive that and the Gronwall lemma we have the following chain of inequalities

$$\mathbb{E} \sup_{s \in [0, t]} \|\mathbf{u}(s)\|_{\frac{1}{4}}^p + 2\mathbb{E} \left(\int_0^t \|\mathbf{u}(t)\|_{\frac{3}{4}}^2 dt \right)^{\frac{p}{2}} \leq \mathbb{E} \|\mathbf{u}(0)\|_{\frac{1}{4}}^p + C^2(1 + \mathbb{E}|\mathbf{u}_0|^{2p})^2 + CT + \mathbb{E} \int_0^t \|\mathbf{u}(s)\|_{\frac{1}{4}}^p ds.$$

Now, we apply Gronwall's inequality for the function $\mathbb{E} \sup_{s \in [0, t]} \|\mathbf{u}(s)\|_{\frac{1}{4}}^p$ to infer that

$$\mathbb{E} \sup_{s \in [0, t]} \|\mathbf{u}(s)\|_{\frac{1}{4}}^p \leq C(1 + \mathbb{E} \|\mathbf{u}(0)\|_{\frac{1}{4}}^p + (\mathbb{E}|\mathbf{u}_0|^{2p})^2),$$

from which along with the former estimate we readily complete the proof of proposition. \square

An \mathbb{F} -adapted process $\mathbf{u} \in C([0, T]; H)$ \mathbb{P} -a.s. is called a mild solution to (1.1) if for every $t \in [0, T]$ and \mathbb{P} -a.s.,

$$\mathbf{u}(t) = e^{-tA} \mathbf{u}_0 + \int_0^t e^{-(t-r)A} \mathbf{B}(\mathbf{u}(r), \mathbf{u}(r)) dr + \int_0^t e^{-(t-r)A} G(\mathbf{u}(r)) dW(r). \quad (2.21)$$

Remark 2.4. Observe that if $\mathbf{u} \in L^2(0, T; V) \cap C([0, T], H)$ is a mild solution to (1.1), then for any $t > s \geq 0$,

$$\mathbf{u}(t) = e^{-(t-s)A} \mathbf{u}(s) + \int_s^t e^{-(t-r)A} \mathbf{B}(\mathbf{u}(r), \mathbf{u}(r)) dr + \int_s^t e^{-(t-r)A} G(\mathbf{u}(r)) dW(r).$$

In fact, we have

$$\begin{aligned} \mathbf{u}(t) &= e^{-(t-s)A} \left(e^{-sA} \mathbf{u}_0 + \int_0^s e^{-(s-r)A} \mathbf{B}(\mathbf{u}(r), \mathbf{u}(r)) dr + \int_0^s e^{-(s-r)A} G(\mathbf{u}(r)) dW(r) \right) \\ &\quad + \int_s^t e^{-(t-r)A} \mathbf{B}(\mathbf{u}(r), \mathbf{u}(r)) dr + \int_s^t e^{-(t-r)A} G(\mathbf{u}(r)) dW(r) \\ &= e^{-(t-s)A} \mathbf{u}(s) + \int_s^t e^{-(t-r)A} \mathbf{B}(\mathbf{u}(r), \mathbf{u}(r)) dr + \int_s^t e^{-(t-r)A} G(\mathbf{u}(r)) dW(r). \end{aligned}$$

We will use this remark later on to prove a very important lemma for our analysis, see Lemma 4.1.

2.3. The numerical scheme and the main result

Let N be a positive integer, $H_N \subset H$ the linear space spanned by $\{\psi_n; n = 1, \dots, N\}$, and $\pi_N : H \rightarrow H_N$ the orthogonal projection of H to the finite dimensional subspace H_N . The projection of \mathbf{u} by π_N is denoted by

$$\mathbf{u}^N := \pi_N \mathbf{u} = \sum_{n=1}^N (\psi_n, \mathbf{u}) \psi_n, \quad (2.22)$$

for $\mathbf{u} \in H$ and $N \in \mathbb{N}$. The Galerkin projection of the SPDEs (1.1) reads

$$d\mathbf{u}^N = [\pi_N A \mathbf{u}^N + \pi_N \mathbf{B}(\mathbf{u}^N, \mathbf{u}^N)] dt + \pi_N G(\mathbf{u}^N) dW(t), \quad \mathbf{u}^N(0) = \pi_N \mathbf{u}_0. \quad (2.23)$$

Due to the assumption (B1)-(B3) and (G), we can use Proposition 2.3 to prove that (2.23) has global weak solution.

To derive an approximation of the exact solution \mathbf{u} of (1.1) we construct an approximation \mathbf{U}^N of the Galerkin solution. To this end, let M be a positive integer and $I_M = ([t_m, t_{m+1}])_{m=0}^M$ an equidistant grid of mesh-size $k = t_{m+1} - t_m$ covering $[0, T]$. Now, for any $j \in \{0, \dots, M-1\}$ we look for a sequence of \mathbb{F} -adapted random variables $\mathbf{U}^j \in \mathbf{H}_N$, $j = 0, 2, \dots, M$ such that for any $w \in \mathbf{V}$

$$\begin{aligned} \mathbf{U}^0 &= \pi_N \mathbf{u}_0, \\ \langle \mathbf{U}^{j+1} - \mathbf{U}^j + k[\pi_N \mathbf{A} \mathbf{U}^{j+1} + \pi_N \mathbf{B}(\mathbf{U}^j, \mathbf{U}^{j+1})], w \rangle &= \langle \pi_N \mathbf{G}(\mathbf{U}^j) \Delta_{j+1} W, w \rangle, \end{aligned} \quad (2.24)$$

where $\Delta_{j+1} W := W(t_{j+1}) - W(t_j)$, $j \in \{0, \dots, M-1\}$, is an independently and identically distributed random variables. We will justify in the following proposition that for a given $\mathbf{U}_0 = \pi_N \mathbf{u}_0$ the numerical scheme (2.24) admits at least one solution $\mathbf{U}^{j+1} \in \mathbf{H}_N$, $j \in \{0, \dots, M-1\}$ and that (2.24) is stable in \mathbf{H} and $D(\mathbf{A}^{\frac{1}{4}})$.

Proposition 2.5. *Let the assumptions (B1)-(B3) and (G) hold. Let N and M be two fixed positive integers and $\mathbf{u}_0 \in \mathbf{L}^{2^p}(\Omega; \mathbf{H})$ for any integer $p \in [2, 4]$. Then, for any $j \in \{0, \dots, M-1\}$ there exists at least a \mathcal{F}_{t_j} -measurable random variable $\mathbf{U}^j \in \mathbf{H}_N$ satisfying (2.24). Moreover, there exists a constant $C > 0$ (dependeing only on T and $\text{Tr } Q$) such that*

$$\mathbb{E} \max_{0 \leq m \leq M} |\mathbf{U}^m|^2 + \sum_{j=0}^{M-1} |\mathbf{U}^{j+1} - \mathbf{U}^j|^2 + 2k \mathbb{E} \sum_{j=1}^M \|\mathbf{U}^j\|_{\frac{1}{2}}^2 \leq C(\mathbb{E}|\mathbf{u}_0|^2 + 1), \quad (2.25)$$

$$\mathbb{E} \left[\max_{1 \leq m \leq M} |\mathbf{U}^m|^{2^p} + k \sum_{m=1}^M |\mathbf{U}^m|^{2^{p-1}} \|\mathbf{U}^m\|_{\frac{1}{2}}^2 \right] \leq C(1 + \mathbb{E}|\mathbf{u}_0|^{2^{p-1}}) \quad (2.26)$$

$$\mathbb{E} \left[k \sum_{m=1}^M \|\mathbf{U}^m\|_{\frac{1}{2}}^2 \right]^{2^{p-1}} \leq C(1 + \mathbb{E}|\mathbf{u}_0|^{2^p}). \quad (2.27)$$

Furthermore, if $\mathbf{u}_0 \in \mathbf{L}^4(\Omega, D(\mathbf{A}^{\frac{1}{4}}))$, then exists a constant $C > 0$ such that

$$\mathbb{E} \max_{0 \leq m \leq M} \|\mathbf{U}^m\|_{\frac{1}{4}}^2 + \mathbb{E} \sum_{j=0}^{m-1} \|\mathbf{U}^{j+1} - \mathbf{U}^j\|_{\frac{1}{4}}^2 + k \mathbb{E} \sum_{j=1}^M \|\mathbf{U}^j\|_{\frac{3}{4}}^2 \leq C, \quad (2.28)$$

$$\mathbb{E} \max_{0 \leq m \leq M} \|\mathbf{U}^m\|_{\frac{1}{4}}^4 + \mathbb{E} \left(\sum_{j=0}^{m-1} \|\mathbf{U}^{j+1} - \mathbf{U}^j\|_{\frac{1}{4}}^2 \right)^2 + k^2 \mathbb{E} \left(\sum_{j=1}^M \|\mathbf{U}^j\|_{\frac{3}{4}}^2 \right)^2 \leq C \quad (2.29)$$

Proof. The detailed proofs of the existence, measurability and the estimates (2.28) and (2.29) will be given in Section 3. Thanks to the assumption (B2), the proof of the inequalities (2.25)-(2.27) is very similar to the proof of [5], so we omit it. \square

Now, we proceed to the statement of the main result of this paper.

Theorem 2.6. *Let the assumptions (B1)-(B3) and (G) hold and assume that $\mathbf{u}_0 \in \mathbf{L}^{16}(\Omega; \mathbf{H}) \cap \mathbf{L}^4(\Omega; \mathbf{V}_{\frac{1}{4}})$. Then for any $\beta \in [0, \frac{1}{4})$, there exists a constant $k_0 > 0$ such that for any small number $\varepsilon > 0$ we have*

$$\mathbb{E} \left(\mathbf{1}_{\Omega_k} \|\mathbf{u}(t_j) - \mathbf{U}^j\|_{\beta}^2 \right) + 2k \mathbb{E} \left(\mathbf{1}_{\Omega_k} \sum_{j=1}^M \|\mathbf{u}(t_j) - \mathbf{U}^j\|_{\frac{1}{2}+\beta}^2 \right) < k_0 k^{-2\varepsilon} [k^{2(\frac{1}{4}-\beta)} + \lambda_N^{-2(\frac{1}{4}-\beta)}], \quad (2.30)$$

where the set Ω_k is defined by

$$\Omega_k = \left\{ \omega \in \Omega : \sup_{t \in [0, T]} \|\mathbf{u}(t, \omega)\|_{\frac{1}{4}}^2 < \log k^{-\varepsilon}, \max_{0 \leq j \leq M} \|\mathbf{U}^j(\omega)\|_{\frac{1}{4}}^2 < \log k^{-\varepsilon} \right\}.$$

Proof. The proof of this theorem will be given in Section 4. \square

Remark 2.7. Note that owing to (2.13) and (2.29) and the Markov inequality it is not difficult to prove that the set Ω_k satisfies

$$\lim_{k \searrow 0} \mathbb{P}[\Omega \setminus \Omega_k] = 0.$$

Corollary 2.8. *If all the assumptions of Theorem 2.6 are satisfied, then the solution $\{\mathbf{U}^j; j = 1, 2, \dots, M\}$ of the numerical scheme (2.24) converges in probability in the Hilbert space \mathbb{V}_β , $\beta \in [0, \frac{1}{4})$. More precisely, for any small number $\varepsilon > 0$, any $\theta_0 \in (0, \frac{1}{4} - \beta - \varepsilon)$ and $\theta_1 \in (0, \frac{1}{4} - \beta)$ we have*

$$\lim_{\Theta \nearrow \infty} \lim_{k \searrow 0} \lim_{N \nearrow \infty} \mathbb{P} \left(\|\mathbf{u}(t_j) - \mathbf{U}^j\|_\beta + k^{\frac{1}{2}} \left(\sum_{j=1}^M \|\mathbf{u}(t_j) - \mathbf{U}^j\|_{\frac{1}{2}+\beta}^2 \right)^{\frac{1}{2}} \geq \Theta[k^{\theta_0} + \Lambda_N^{-\theta_1}] \right) = 0. \quad (2.31)$$

Proof. To shorten notation let us set $\mathbf{e}^j := \mathbf{u}(t_j) - \mathbf{U}^j$ and

$$\Omega_{k,N}^\Theta = \{\omega \in \Omega; \|\mathbf{e}^j\|_\beta^2 + k \sum_{j=1}^M \|\mathbf{e}^j\|_{\frac{1}{2}+\beta}^2 \geq \Theta[k^{\theta_0} + \Lambda_N^{-\theta_1}]\},$$

for any positive numbers M and k . Let Ω_k be as in the statement of Theorem 2.6. Owing to (2.30), (2.13), (2.29) and the Chebychev-Markov, we can find a constant $\tilde{C}_5 > 0$ such that

$$\begin{aligned} \mathbb{P}(\Omega_{k,N}^\Theta) &= \mathbb{P}(\Omega_{k,N}^\Theta \cap \Omega_k) + \mathbb{P}(\Omega_{k,N}^\Theta \cap \Omega_k^c) \\ &\leq \mathbb{P}(\Omega_{k,N}^\Theta \cap \Omega_k) + \mathbb{P}(\Omega_k^c) \\ &\leq \frac{k_0}{\Theta} k^{2(\frac{1}{4}-\beta)-2\varepsilon-2\theta_0} + \frac{k_0}{\Theta} k^{-2\varepsilon} \lambda_N^{-2(\frac{1}{4}-\beta)+2\theta_1} + \frac{\tilde{C}_5}{\log k^{-\varepsilon}}. \end{aligned}$$

Letting $N \nearrow \infty$, then $k \searrow 0$ and finally $\Theta \nearrow \infty$ in the last line we easily conclude the proof of the corollary. \square

To close this section let us make some few remarks. Instead of the scheme (2.24) we could also use a fully-implicit scheme. More precisely, for any $j \in \{0, \dots, M-1\}$ we look for a \mathcal{F}_{t_j} -measurable random variable $\mathcal{U}^j \in \mathbb{H}_N$ such that for any $w \in \mathbb{V}$

$$\begin{aligned} \mathcal{U}^0 &= \pi_N \mathbf{u}_0, \\ \langle \mathcal{U}^{j+1} - \mathcal{U}^j + k[\pi_N \mathbf{A} \mathcal{U}^{j+1} + \pi_N \mathbf{B}(\mathcal{U}^{j+1}, \mathcal{U}^{j+1})], w \rangle &= \langle \pi_N \mathbf{G}(\mathcal{U}^j) \Delta_{j+1} W, w \rangle, \end{aligned} \quad (2.32)$$

where $\Delta_{j+1} W := W(t_{j+1}) - W(t_j)$, $j \in \{0, \dots, M-1\}$. We have the following theorem

Theorem 2.9. *Let the assumptions (B1)-(B3) and (G) hold and assume that $\mathbf{u}_0 \in L^{16}(\Omega; \mathbb{H}) \cap L^4(\Omega; \mathbb{V}_{\frac{1}{4}})$. Let N and M be two fixed positive integers. Then,*

(a) *for any $j \in \{0, \dots, M-1\}$ there exists a unique \mathcal{F}_{t_j} -measurable random variable $\mathcal{U}^j \in \mathbb{H}_N$ satisfying (2.32) and the estimates (2.25) and (2.29).*

(b) *For any $\beta \in [0, \frac{1}{4})$ there exists a constant $k_0 > 0$ such that for any small number $\varepsilon > 0$ we have*

$$\mathbb{E} \left(\mathbf{1}_{\Omega_k} \|\mathbf{u}(t_j) - \mathcal{U}^j\|_\beta^2 \right) + 2k \mathbb{E} \left(\mathbf{1}_{\Omega_k} \sum_{j=1}^M \|\mathbf{u}(t_j) - \mathcal{U}^j\|_{\frac{1}{2}+\beta}^2 \right) < k_0 k^{-2\varepsilon} [k^{2(\frac{1}{4}-\beta)} + \lambda_N^{-2(\frac{1}{4}-\beta)}], \quad (2.33)$$

where

$$\Omega_k = \left\{ \omega : \sup_{t \in [0, T]} \|\mathbf{u}(t, \omega)\|_{\frac{1}{4}}^2 < \log k^{-\varepsilon}, \max_{0 \leq j \leq M} \|\mathcal{U}^j(\omega)\|_{\frac{1}{4}}^2 < \log k^{-\varepsilon} \right\}.$$

(c) Moreover, for any small number $\varepsilon > 0$, any $\theta_0 \in \left(0, \frac{1}{4} - \beta - \varepsilon\right)$ and $\theta_1 \in (0, \frac{1}{4} - \beta)$

$$\lim_{\Theta \nearrow \infty} \lim_{k \searrow 0} \lim_{N \nearrow \infty} \mathbb{P} \left(\|\mathbf{u}(t_j) - \mathcal{U}^j\|_\beta + k^{\frac{1}{2}} \left(\sum_{j=1}^M \|\mathbf{u}(t_j) - \mathcal{U}^j\|_{\frac{2}{2+\beta}}^2 \right)^{\frac{1}{2}} \geq \Theta[k^{\theta_0} + \Lambda_N^{-\theta_1}] \right) = 0. \quad (2.34)$$

Proof. The arguments for the proof of this theorem are very similar to those of the proofs of Proposition 2.5, Theorem 2.6 and Corollary 2.8, thus we omit them. \square

3. Existence and stability analysis of the scheme: Proof of Proposition 2.5

In this section we will show that for any $j \in \{0, \dots, M-1\}$ the numerical scheme (2.24) admits at least one solution $\mathbf{U}^j \in \mathbf{H}_N$. We will also show that (2.24) is stable in $D(A^{\frac{1}{4}})$ provided that $\mathbf{u}_0 \in L^4(\Omega; D(A^{\frac{1}{4}})) \cap L^{16}(\Omega; \mathbf{H})$. The precise statement of these facts was already done in Proposition 2.5, thus we proceed directly to their proofs.

Proof of Proposition 2.5. As we mentioned in Subsection 2.3 we will only prove the existence, measurability and the estimates (2.28) and (2.29). The proof of the inequalities (2.25)-(2.27) will be omitted because it is very similar to the proof of [5] (see also [6]).

Proof of the existence. We first establish that for any $j \in \{0, \dots, M-1\}$ there exists $\mathbf{U}^j \in \mathbf{H}_N$ satisfying the numerical scheme (2.24). To this end, let us fix $\omega \in \Omega$ and for a given $\mathbf{U}^j \in \mathbf{H}_N$ consider the map $\Lambda_\omega^j : \mathbf{H}_N \rightarrow \mathbf{H}_N$ defined by

$$(\Lambda_\omega^j(\mathbf{v}), \psi) = (\mathbf{v} - \mathbf{U}^j(\omega), \psi) + k(\mathbf{A}\mathbf{v} + \pi_N \mathbf{B}(\mathbf{U}^j(\omega), \mathbf{v}), \psi) + (G(\mathbf{U}^j(\omega))\Delta_{j+1}W(\omega), \psi)$$

for any $\psi \in \mathbf{H}_N$. Note that since $\mathbf{H}_N \subset D(\mathbf{A})$ the map Λ_ω^j is well-defined. From assumptions (B1) and (G) and the linearity of \mathbf{A} it is clear that for given \mathbf{U}^j the map Λ_ω^j is continuous. Furthermore, using Hölder's inequality the fact that $\lambda_1|\psi|^2 \leq \|\psi\|_{\frac{2}{1}}^2$, $\psi \in \mathbf{V}$ and assumptions (B2) and (G) we derive that

$$\begin{aligned} (\Lambda_\omega^j \mathbf{v}, \mathbf{v}) &\geq |\mathbf{v}|^2 \left(\lambda_1 k + \frac{1}{2} - \frac{k}{2} \right) - \frac{|\mathbf{U}^j(\omega)|^2}{2} \left(1 + \|\Delta_{j+1}W(\omega)\|_{\mathcal{H}}^2 C_2^2 \right) - \frac{1}{2} \|\Delta_{j+1}W(\omega)\|_{\mathcal{H}}^2 C_2^2 \\ &\geq \gamma |\mathbf{v}|^2 - \Gamma_\omega^j. \end{aligned}$$

Since $k < 1$ and, by Assumption (N), $\|\Delta_{j+1}W\|_{\mathcal{H}}^2 < \infty$, the constant γ is positive and $\mu_j = \sqrt{\frac{\Gamma_\omega^j}{\gamma}} < \infty$ whenever $|\mathbf{U}^j|^2 < \infty$. Thus, we have $(\Lambda_\omega^j \mathbf{v}, \mathbf{v}) \geq 0$ for any $\mathbf{v} \in \mathcal{H}_N^j(\omega) := \{\psi \in \mathbf{H}_N; |\psi| = R\mu_j\}$ where $R > 1$ is an arbitrary constant. Since $\mathbf{U}^0 = \pi_N \mathbf{u}_0$ is given, we can conclude from the above observations and Brouwer fixed point theorem (see, for instance, [14]) that there exists at least one $\mathbf{U}^1 \in \mathbf{H}_N$ satisfying

$$\Lambda_\omega^0(\mathbf{U}^1) = 0 \text{ and } |\mathbf{U}^1| \leq R\mu_0.$$

In a similar way, assuming that $\mathbf{U}^j \in \mathbf{H}_N$ we infer that there exists at least one $\mathbf{U}^{j+1} \in \mathbf{H}_N$ such that

$$\Lambda_\omega^j(\mathbf{U}^{j+1}) = 0 \text{ and } |\mathbf{U}^{j+1}| \leq R\mu_j.$$

Therefore, we have just prove by induction that given $\mathbf{U}^0 \in \mathbf{H}_N$ and a \mathcal{H} -valued Wiener process W , for each j there exists a sequence $\{\mathbf{U}^j; j = 1, \dots, M\} \subset \mathbf{H}_N$ satisfying the algorithm (2.24).

Proof of the measurability. In order to prove the \mathcal{F}_{t_j} -measurability of \mathbf{U}^j it is sufficient to show that for each $j \in \{1, \dots, M\}$ one can find a Borel measurable map $\mathcal{E}_j : \mathbf{H}_N \times \mathcal{H} \rightarrow \mathbf{H}_N$ such that $\mathbf{U}^j = \mathcal{E}_j(\mathbf{U}^{j-1}, \Delta_j W)$. In fact, if such claim is true then by exploiting the \mathcal{F}_{t_j} -measurability of $\Delta_j W$ one

can argue by induction and show that if U^0 is \mathcal{F}_0 -measurable then $\mathcal{E}_j(U^{j-1}, \Delta_j W)$, hence U^j , is \mathcal{F}_{t_j} -measurable. Thus, it remains to prove the existence of \mathcal{E}_j . For this purpose we will closely follow [15]. Let $\mathcal{P}(H_N)$ be the set of subsets of H_N and consider a multivalued map $\mathcal{E}_{j+1}^S : H_N \times \mathcal{H} \rightarrow \mathcal{P}(H_N)$ such that for each (U^j, η_{j+1}) , $\mathcal{E}_{j+1}^S(U^j, \eta_{j+1})$ denotes the set of solution U^{j+1} of (2.24). From the existence result above we deduce that \mathcal{E}_{j+1}^S maps $H_N \times \mathcal{H}$ to nonempty closed subsets of H_N . Furthermore, since we are in finite dimensional space H_N , we can prove, by using the assumptions (B1) and (G) and the sequential characterization of the closed graph theorem, that the graph of \mathcal{E}_{j+1}^S is closed. From these last two facts and [16, Theorem 3.1] we can find a univocal map $\mathcal{E}_{j+1} : H_N \times \mathcal{H} \rightarrow H_N$ such that $\mathcal{E}_j(U^j, \eta_{j+1}) \in \mathcal{E}_{j+1}^S(U^j, \eta_{j+1})$ and \mathcal{E}_j is measurable when $H_N \times \mathcal{H}$ and H_N are equipped with their respective Borel σ -algebra. This completes the proof of the measurability of the solutions of (2.24). *Proof of (2.25)-(2.27).* Thanks to the assumption (B2), the proof of the inequalities (2.25)-(2.27) is very similar to the proof of [5], so we omit it and we directly proceed to the proof of the estimates (2.28) and (2.29).

Proof of (2.28). Taking $w = 2A^{\frac{1}{2}}U^{j+1}$ in (2.24), using the Cauchy-Schwarz inequality and the identity

$$((v - x, 2v)) = \|v\|^2 - \|x\|^2 + \|v - x\|^2, \quad (v, x \text{ are elements of a Hilbert space with norm } \|\cdot\|) \quad (3.1)$$

yield

$$\begin{aligned} & \|U^{j+1}\|_{\frac{1}{4}}^2 - \|U^j\|_{\frac{1}{4}}^2 + \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 + 2k\|U^{j+1}\|_{\frac{3}{4}}^2 \\ & \leq 2k|B(U^j, U^{j+1})|\|U^{j+1}\|_{\frac{1}{2}} + 2\|G(U^j)\Delta_{j+1}W\|_{\frac{1}{4}}\|U^{j+1} - U^j\|_{\frac{1}{4}} \\ & \quad + 2\langle A^{\frac{1}{4}}G(U^j)\Delta_{j+1}W, A^{\frac{1}{4}}U^j \rangle. \end{aligned} \quad (3.2)$$

Using Assumption (B1)', the complex interpolation inequality in [10, Theorem 1.9.3, pp 59] and the Young inequality

$$2|B(U^j, U^{j+1})|\|U^{j+1}\|_{\frac{1}{2}} \leq C|U^j|^4\|U^{j+1}\|_{\frac{1}{4}}^2 + \|U^{j+1}\|_{\frac{3}{4}}^2 \quad (3.3)$$

Using the continuous embedding $V_{\frac{1}{2}} \subset V_{\frac{1}{4}}$ we obtain

$$2|B(U^j, U^{j+1})|\|U^{j+1}\|_{\frac{1}{2}} \leq C|U^j|^4\|U^{j+1}\|_{\frac{1}{2}}^2 + \|U^{j+1}\|_{\frac{3}{4}}^2,$$

which implies that

$$\begin{aligned} & \|U^{j+1}\|_{\frac{1}{4}}^2 - \|U^j\|_{\frac{1}{4}}^2 + \frac{1}{2}\|U^{j+1} - U^j\|_{\frac{1}{4}}^2 + 2k\|U^{j+1}\|_{\frac{3}{4}}^2 \leq 2Ck|U^j|^4\|U^{j+1}\|_{\frac{1}{2}}^2 + \|G(U^j)\Delta_{j+1}W\|_{\frac{1}{4}}^2 \\ & \quad + 2\langle A^{\frac{1}{4}}G(U^j)\Delta_{j+1}W, A^{\frac{1}{4}}U^j \rangle. \end{aligned} \quad (3.4)$$

Since U^j is a constant adapted and hence progressively measurable, it is not difficult to prove that

$$2\mathbb{E}\langle A^{\frac{1}{4}}G(U^j)\Delta_{j+1}W, A^{\frac{1}{4}}U^j \rangle = 0.$$

Using (2.26) and (2.27) with $p = 2$ and $p = 3$ respectively, we easily prove that there exists a constant $C > 0$, depending only on T , and the norm of $u_0 \in L^8(\Omega, H)$, such that

$$k\mathbb{E} \left(\sum_{j=0}^{M-1} |U^j|^4 \|U^{j+1}\|_{\frac{1}{2}}^2 \right) \leq \left(\mathbb{E} \max_{1 \leq m \leq M} |U^m|^8 \right)^{\frac{1}{2}} \left(\mathbb{E} \left(k \sum_{j=1}^M \|U^j\|_{\frac{1}{2}}^2 \right)^2 \right)^{\frac{1}{2}} \leq C(1 + \mathbb{E}|u_0|^8)^2. \quad (3.5)$$

Now, since U^j is \mathcal{F}_{t_j} -measurable and $\Delta_{j+1}W$ is independent of \mathcal{F}_{t_j} , we infer that there exists a constant

$C > 0$ such that for any $j \in \{1, \dots, M\}$

$$\begin{aligned}
\mathbb{E} \left(\|G(U^j) \Delta_{j+1} W\|_{\frac{1}{4}}^2 \right) &\leq \mathbb{E} \left(\mathbb{E} \left(\|G(U^j)\|_{\mathcal{L}(\mathcal{H}, V_{\frac{1}{4}})}^2 \|\Delta_{j+1} W\|_{\mathcal{H}}^2 | \mathcal{F}_{t_j} \right) \right), \\
&= m \mathbb{E} \left(\|G(U^j)\|_{\mathcal{L}(\mathcal{H}, V_{\frac{1}{4}})}^2 \mathbb{E} \left(\|\Delta_{j+1} W\|_{\mathcal{H}}^2 | \mathcal{F}_{t_j} \right) \right), \\
&\leq Ck \text{ (tr} Q)^{\frac{1}{2}} (1 + \mathbb{E} \|U^j\|_{\frac{1}{4}}^2),
\end{aligned} \tag{3.6}$$

where (2.7) and Assumption (G) along with Remark 2.1-(b) were used to derive the last line of the above chain of inequalities.

Now taking the mathematical expectation in (3.4), summing both sides of the resulting equations from $j = 1$ to $m - 1$ and using the last three observations imply

$$\begin{aligned}
\max_{1 \leq m \leq M} \mathbb{E} \|U^m\|_{\frac{1}{4}}^2 + \frac{1}{2} \mathbb{E} \left(\sum_{j=0}^{M-1} \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 \right) + 2k \mathbb{E} \sum_{j=1}^M \|U^j\|_{\frac{3}{4}}^2 \\
\leq CT + \mathbb{E} \|\mathbf{u}_0\|_{\frac{1}{4}}^2 + C \text{Tr } Qk \sum_{m=1}^M \max_{1 \leq j \leq m} \mathbb{E} \|U^j\|_{\frac{1}{4}}^2,
\end{aligned}$$

from which along with the discrete Gronwall lemma we infer that there exists a constant $C > 0$ such that

$$\max_{1 \leq m \leq M} \mathbb{E} \|U^m\|_{\frac{1}{4}}^2 + \frac{1}{2} \mathbb{E} \left(\sum_{j=0}^{M-1} \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 \right) + 2k \mathbb{E} \sum_{j=1}^M \|U^j\|_{\frac{3}{4}}^2 \leq C(1 + \mathbb{E} \|\mathbf{u}_0\|_{\frac{1}{4}}^2 + [\mathbb{E} |\mathbf{u}_0|^8]^2). \tag{3.7}$$

Note that from (3.4) we can derive that there exists a constant $C > 0$ such that

$$\begin{aligned}
\mathbb{E} \max_{1 \leq m \leq M} \|U^m\|_{\frac{1}{4}}^2 &\leq \mathbb{E} \|\mathbf{u}_0\|_{\frac{1}{4}}^2 + Ck \mathbb{E} \sum_{j=0}^{M-1} |U^j|^4 \|U^{j+1}\|_{\frac{1}{2}}^2 + \mathbb{E} \sum_{j=0}^{M-1} \|G(U^j) \Delta_{j+1} W\|_{\frac{1}{4}}^2 \\
&\quad + 2 \mathbb{E} \max_{1 \leq m \leq M} \sum_{j=0}^{m-1} \langle A^{\frac{1}{4}} G(U^j) \Delta_{j+1} W, A^{\frac{1}{4}} U^j \rangle \\
&=: \sum_{i=1}^4 I_i.
\end{aligned}$$

Arguing as in [6, proof of (3.9)] we can establish that

$$I_4 \leq \frac{1}{2} \mathbb{E} \|\mathbf{u}_0\|_{\frac{1}{4}}^2 + \frac{1}{2} \mathbb{E} \max_{1 \leq m \leq M} \|U^m\|_{\frac{1}{4}}^2 + Ck \sum_{j=0}^{M-1} \mathbb{E} \|U^j\|_{\frac{1}{4}}^2,$$

which altogether with (3.7) yields that

$$I_4 \leq \frac{1}{2} \mathbb{E} \max_{1 \leq m \leq M} \|U^m\|_{\frac{1}{4}}^2 + C(1 + \mathbb{E} \|\mathbf{u}_0\|_{\frac{1}{4}}^2).$$

Using the same idea as in the proof of (3.6) and using (3.7) we infer that

$$I_3 \leq C(1 + \mathbb{E} \|\mathbf{u}_0\|_{\frac{1}{4}}^2).$$

Using these two estimates and the inequality (3.5) we derive that there exists a constant $C > 0$ such that

$$\mathbb{E} \max_{1 \leq m \leq M} \|U^m\|_{\frac{1}{4}}^2 \leq C(1 + \mathbb{E} \|\mathbf{u}_0\|_{\frac{1}{4}}^2 + [\mathbb{E} |\mathbf{u}_0|^8]^2),$$

which along with (3.7) completes the proof of (2.28).

Now, we continue with the derivation of an estimate of $\mathbb{E} \max_{0 \leq j \leq M} \|U^j\|_{\frac{1}{4}}^4$. Multiplying (3.2) by $\|U^{j+1}\|_{\frac{1}{4}}^2$ and using identity (3.1) and then summing both sides of the resulting equation from $j = 0, \dots, m-1$ implies

$$\begin{aligned}
\frac{1}{2} \|U^m\|_{\frac{1}{4}}^4 + \frac{1}{2} \sum_{j=0}^{m-1} \left| \|U^{j+1}\|_{\frac{1}{4}}^2 - \|U^j\|_{\frac{1}{4}}^2 \right|^2 + \sum_{j=0}^{m-1} \|U^{j+1}\|_{\frac{1}{4}}^2 \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 + 2k \sum_{j=0}^{m-1} \|U^{j+1}\|_{\frac{1}{4}}^2 \|U^{j+1}\|_{\frac{3}{4}}^2 \\
\leq \frac{1}{2} \|\mathbf{u}_0\|_{\frac{1}{4}}^4 + Ck \sum_{j=0}^{m-1} |\mathbf{B}(U^j, U^{j+1})|^2 \|U^{j+1}\|_{\frac{1}{2}}^2 \|U^{j+1}\|_{\frac{1}{4}}^2 \\
+ 2 \sum_{j=0}^{m-1} \langle A^{\frac{1}{4}} G(U^j) \Delta_{j+1} W, A^{\frac{1}{4}} [U^{j+1} - U^j] \rangle \|U^{j+1}\|_{\frac{1}{4}}^2 \quad (3.8) \\
+ 2 \sum_{j=0}^{m-1} \langle A^{\frac{1}{4}} G(U^j) \Delta_{j+1} W, A^{\frac{1}{4}} U^j \rangle \|U^{j+1}\|_{\frac{1}{4}}^2 \\
=: \frac{1}{2} \|\mathbf{u}_0\|_{\frac{1}{4}}^4 + J_1 + J_2 + J_3.
\end{aligned}$$

Thanks to the estimate (3.3) we can estimate J_1 as follows

$$\mathbb{E} J_1 \leq Ck \mathbb{E} \sum_{j=0}^{m-1} |U^j|^4 \|U^{j+1}\|_{\frac{1}{4}}^4 + k \mathbb{E} \sum_{j=0}^{m-1} \|U^{j+1}\|_{\frac{1}{4}}^2 \|U^{j+1}\|_{\frac{3}{4}}^2 =: J_{1,1} + J_{1,2}.$$

Since the second term $J_{1,2}$ can be absorbed in the LHS we will focus on estimating the second term $J_{1,1}$. We have

$$\begin{aligned}
J_{1,1} &\leq Ck \sum_{j=0}^{M-1} |U^j|^4 |U^{j+1}|^2 \|U^{j+1}\|_{\frac{1}{2}}^2 \\
&\leq C \left(\mathbb{E} \max_{0 \leq j \leq M-1} [|U^j|^8 |U^{j+1}|^4] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[k \sum_{j=1}^M \|U^j\|_{\frac{1}{2}}^2 \right]^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\mathbb{E} \left[\max_{0 \leq j \leq M-1} |U^j|^{12} \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[k \sum_{j=1}^M \|U^j\|_{\frac{1}{2}}^2 \right]^2 \right)^{\frac{1}{2}} \\
&\leq C(1 + \mathbb{E} |\mathbf{u}_0|^{16}),
\end{aligned}$$

where (2.26) and (2.27) are used to obtain the last line. Hence,

$$\mathbb{E} J_1 \leq C(1 + \mathbb{E} |\mathbf{u}_0|^{16}) + \mathbb{E} k \sum_{j=0}^{m-1} \|U^{j+1}\|^2 - \frac{1}{4} \|U^{j+1}\|_{\frac{3}{4}}^2$$

Now we estimate J_2 as follows

$$\begin{aligned}
\mathbb{E} J_2 &\leq C \mathbb{E} \sum_{j=0}^{m-1} \|G(U^j) \Delta_{j+1} W\|_{\frac{1}{4}}^2 \left(\|U^{j+1}\|_{\frac{1}{4}}^2 - \|U^j\|_{\frac{1}{4}}^2 + \|U^j\|_{\frac{1}{4}}^2 \right) + \frac{1}{2} \mathbb{E} \sum_{j=0}^{m-1} \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 \|U^{j+1}\|_{\frac{1}{4}}^2 \\
&\leq C \mathbb{E} \sum_{j=0}^{m-1} \|G(U^j) \Delta_{j+1} W\|_{\frac{1}{4}}^4 + C \mathbb{E} \sum_{j=0}^{m-1} \|G(U^j) \Delta_{j+1} W\|_{\frac{1}{4}}^2 \|U^j\|_{\frac{1}{4}}^2 + \frac{1}{8} \mathbb{E} \sum_{j=0}^{m-1} \left| \|U^{j+1}\|_{\frac{1}{4}}^2 - \|U^j\|_{\frac{1}{4}}^2 \right|^2 \\
&\quad + \frac{1}{2} \mathbb{E} \sum_{j=0}^{m-1} \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 \|U^{j+1}\|_{\frac{1}{4}}^2.
\end{aligned}$$

As long as J_3 is concerned we have

$$\begin{aligned}
\mathbb{E}J_3 &= 2\mathbb{E} \sum_{j=0}^{m-1} \langle A^{\frac{1}{4}}G(U^j)\Delta_{j+1}W, A^{\frac{1}{4}}U^j \rangle \|U^j\|_{\frac{1}{4}}^2 + 2\mathbb{E} \sum_{j=0}^{m-1} \langle A^{\frac{1}{4}}G(U^j)\Delta_{j+1}W, A^{\frac{1}{4}}U^j \rangle \left(\|U^{j+1}\|^2 \frac{1}{4} - \|U^j\|_{\frac{1}{4}}^2 \right), \\
&= 2\mathbb{E} \sum_{j=0}^{m-1} \langle A^{\frac{1}{4}}G(U^j)\Delta_{j+1}W, A^{\frac{1}{4}}U^j \rangle \left(\|U^{j+1}\|^2 \frac{1}{4} - \|U^j\|_{\frac{1}{4}}^2 \right), \\
&\leq C\mathbb{E} \sum_{j=0}^{m-1} \|A^{\frac{1}{4}}G(U^j)\Delta_{j+1}W\|_{\frac{1}{4}}^2 \|U^j\|_{\frac{1}{4}}^2 + \frac{1}{8}\mathbb{E} \sum_{j=0}^{m-1} \left| \|U^{j+1}\|^2 \frac{1}{4} - \|U^j\|_{\frac{1}{4}}^2 \right|^2
\end{aligned}$$

because for any j

$$\mathbb{E} \langle A^{\frac{1}{4}}G(U^j)\Delta_{j+1}W, A^{\frac{1}{4}}U^j \rangle \|U^j\|_{\frac{1}{4}}^2 = 0.$$

By a similar idea as used to derive (3.6) we can prove that

$$C\mathbb{E} \sum_{j=0}^{m-1} \|G(U^j)\Delta_{j+1}W\|_{\frac{1}{4}}^4 + C\mathbb{E} \sum_{j=0}^{m-1} \|G(U^j)\Delta_{j+1}W\|_{\frac{1}{4}}^2 \|U^j\|_{\frac{1}{4}}^2 \leq C + Ck\mathbb{E} \sum_{j=0}^{m-1} \|U^j\|_{\frac{1}{4}}^4.$$

Thus,

$$\mathbb{E}[J_2 + J_3] \leq C + Ck\mathbb{E} \sum_{j=0}^{m-1} \|U^j\|_{\frac{1}{4}}^4 + \frac{1}{4}\mathbb{E} \sum_{j=0}^{m-1} \left| \|U^{j+1}\|^2 \frac{1}{4} - \|U^j\|_{\frac{1}{4}}^2 \right|^2 + \frac{1}{2}\mathbb{E} \sum_{j=0}^{m-1} \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 \|U^{j+1}\|_{\frac{1}{4}}^2.$$

Taking the mathematical expectation in (3.8) and by plugging the information about $J_i, i = 1, 2, 3$ we infer that

$$\begin{aligned}
&\max_{0 \leq j \leq m-1} \frac{1}{2}\mathbb{E} \|U^j\|_{\frac{1}{4}}^4 + \frac{1}{4}\mathbb{E} \sum_{j=0}^{M-1} \left| \|U^{j+1}\|^2 \frac{1}{4} - \|U^j\|_{\frac{1}{4}}^2 \right|^2 \\
&+ \frac{1}{2}\mathbb{E} \sum_{j=0}^{M-1} \|U^{j+1}\|_{\frac{1}{4}}^2 \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 + k\mathbb{E} \sum_{j=1}^M \|U^j\|_{\frac{1}{4}}^2 \|U^j\|_{\frac{3}{4}}^2 \\
&\leq C(1 + \mathbb{E}|\mathbf{u}_0|^{12} + \|\mathbf{u}_0\|_{\frac{1}{4}}^4) + Ck\mathbb{E} \sum_{j=0}^{m-1} \|U^j\|_{\frac{1}{4}}^4,
\end{aligned}$$

which along with the Gronwall inequality yields

$$\max_{0 \leq j \leq M} \frac{1}{2}\mathbb{E} \|U^j\|_{\frac{1}{4}}^4 \leq C(1 + \mathbb{E}|\mathbf{u}_0|^{12} + \|\mathbf{u}_0\|_{\frac{1}{4}}^4).$$

The latter inequality is used in the former one to derive that

$$\begin{aligned}
&\max_{0 \leq j \leq M} \frac{1}{2}\mathbb{E} \|U^j\|_{\frac{1}{4}}^4 + \frac{1}{4}\mathbb{E} \sum_{j=0}^{M-1} \left| \|U^{j+1}\|^2 \frac{1}{4} - \|U^j\|_{\frac{1}{4}}^2 \right|^2 + \frac{1}{2}\mathbb{E} \sum_{j=0}^{M-1} \|U^{j+1}\|_{\frac{1}{4}}^2 \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 \\
&+ k\mathbb{E} \sum_{j=1}^M \|U^j\|_{\frac{1}{4}}^2 \|U^j\|_{\frac{3}{4}}^2 \leq C(1 + \mathbb{E}|\mathbf{u}_0|^{12} + \|\mathbf{u}_0\|_{\frac{1}{4}}^4). \tag{3.9}
\end{aligned}$$

Now we continue our analysis with the estimation of $\mathbb{E} \max_{0 \leq j \leq M} \|U^j\|_{\frac{1}{4}}^4$. To start with this analysis,

we easily derive from (3.8) the following inequality

$$\begin{aligned} \max_{0 \leq j \leq m-1} \frac{1}{2} \|U^j\|_{\frac{1}{4}}^4 &\leq Ck \sum_{j=0}^{M-1} |U^j|^4 \|U^{j+1}\|^2 \|U^{j+1}\|_{\frac{1}{2}}^2 \\ &\quad + C \sum_{j=0}^{m-1} \left(\|G(U^j)\Delta_{j+1}\|_{\frac{1}{4}}^4 + \|G(U^j)\Delta_{j+1}\|_{\frac{1}{4}}^2 \|U^j\|_{\frac{1}{4}}^2 \right) \\ &\quad + \max_{0 \leq j \leq m-1} \sum_{\ell=0}^{j-1} \langle A^{\frac{1}{4}} G(U^j)\Delta_{j+1}W, A^{\frac{1}{4}} U^j \rangle \|U^j\|_{\frac{1}{4}}^2 =: J_1 + J_2 + J_3. \end{aligned}$$

Arguing as in the proof of (3.6) and using (3.9), the mathematical expectation of $J_1 + J_2$ can be estimated as follows

$$\mathbb{E}(J_1 + J_2) \leq C\mathbb{E}(1 + |\mathbf{u}_0|^{16} + \|\mathbf{u}_0\|_{\frac{1}{4}}^4).$$

The same idea as used in the proof of [6, inequality (3.15)] yields

$$\mathbb{E}J_3 \leq \frac{1}{4}\mathbb{E} \max_{0 \leq j \leq m-1} \|U^j\|_{\frac{1}{4}}^4 + C\mathbb{E}\|\mathbf{u}_0\|_{\frac{1}{4}}^4 + Ck\mathbb{E} \sum_{j=0}^{M-1} \|U^j\|_{\frac{1}{4}}^4,$$

from which altogether with (3.9) we infer that

$$\mathbb{E}J_3 \leq C\mathbb{E}(1 + |\mathbf{u}_0|^{16} + \|\mathbf{u}_0\|_{\frac{1}{4}}^4) + \frac{1}{4}\mathbb{E} \max_{0 \leq j \leq m-1} \|U^j\|_{\frac{1}{4}}^4.$$

Thus, summing up we have shown that there exists a constant $C > 0$ such that

$$\mathbb{E} \max_{0 \leq j \leq M} \|U^j\|_{\frac{1}{4}}^4 \leq C\mathbb{E}(1 + |\mathbf{u}_0|^{16} + \|\mathbf{u}_0\|_{\frac{1}{4}}^4). \quad (3.10)$$

Now we estimate $\mathbb{E} \left(\sum_{j=0}^{M-1} \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 \right)^2 + \mathbb{E} \left(k \sum_{j=1}^M \|U^j\|_{\frac{1}{4}}^2 \right)^2$. To do this we first observe that from (3.4) we obtain

$$\begin{aligned} \left(\frac{1}{2} \sum_{j=0}^{M-1} \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 \right)^2 + \left(2k \sum_{j=0}^{M-1} \|U^{j+1}\|_{\frac{1}{4}}^2 \right)^2 &\leq C \left(k \sum_{j=0}^{M-1} |U^j|^4 \|U^{j+1}\|_{\frac{1}{2}}^2 \right)^2 \\ &\quad + C \left(\sum_{j=0}^{M-1} \|G(U^j)\Delta_{j+1}W\|_{\frac{1}{4}}^2 \right)^2 + C \left(\sum_{j=0}^{M-1} \langle A^{\frac{1}{4}} G(U^j)\Delta_{j+1}W, A^{\frac{1}{4}} U^j \rangle \right)^2. \end{aligned} \quad (3.11)$$

Then, using the same strategies to estimate the J_i -s (or J_i), the sum of the three terms in the right hand side of the above quality can be bounded from above by

$$\left[\mathbb{E} \left(\max_{0 \leq j \leq M} |U^j|^{16} \right) \right]^{\frac{1}{2}} \left[\mathbb{E} \left(k \sum_{j=1}^M \|U^j\|_{\frac{1}{4}}^2 \right)^4 \right]^{\frac{1}{2}} + CMk^2 \sum_{j=0}^M \mathbb{E} \|U^j\|_{\frac{1}{4}}^4 + Ck \sum_{j=0}^M \mathbb{E} \|U^j\|_{\frac{1}{4}}^4,$$

which along with the estimate for $\max_{0 \leq j \leq M} \mathbb{E} \|U^j\|_{\frac{1}{4}}^4$ and the inequalities (2.26) and (2.27) implies that

$$\left(\frac{1}{2} \sum_{j=0}^{M-1} \|U^{j+1} - U^j\|_{\frac{1}{4}}^2 \right)^2 + \left(2k \sum_{j=0}^{M-1} \|U^{j+1}\|_{\frac{1}{4}}^2 \right)^2 \leq \mathbb{E}(1 + |\mathbf{u}_0|^{16} + \|\mathbf{u}_0\|_{\frac{1}{4}}^4), \quad (3.12)$$

which along with (3.10) completes the proof of (2.29) and hence the whole proposition. \square

4. Error analysis of the numerical scheme (2.24): Proof of Theorem 2.6

This section is devoted to the analysis of the error $e_j = \mathbf{u}(t_j) - U^j$ at the time step t_j between the exact solution \mathbf{u} of (1.1) and the approximate solution given by (2.24). Since the precise statement of the convergence rate is already given in Theorem 2.6, thus we proceed directly to the promised proof of Theorem 2.6.

Before giving the proof of Theorem 2.6 we state and prove the following important result.

Lemma 4.1. *Let β be as in Theorem 2.6. Then,*

(i) *there exists a $C_7 > 0$ such that*

$$\mathbb{E} \|\mathbf{u}(t) - \mathbf{u}(s)\|_\beta^2 \leq C_7 [(t-s)^{2-2\beta} + (t-s)^{2(\frac{1}{4}-\beta)} + (t-s)], \quad (4.1)$$

for any $t, s \geq 0$ and $t \neq s$.

(ii) *There also exists a positive constant C_8 such that*

$$\mathbb{E} \int_s^t \|\mathbf{u}(t) - \mathbf{u}(r)\|_{\frac{1}{2}+\beta}^2 dr \leq C_8 \left((t-s)^{\frac{3}{2}-2\beta} + (t-s)^{2(\frac{1}{4}-\beta)} + (t-s)^{2-2\beta} \right), \quad (4.2)$$

for any $t > s \geq 0$.

Proof of Lemma 4.1. As in the statement of the lemma we divide the proof into two parts.

Proof of item (i). Let $t, s \in [0, T]$ such that $t \neq s$. Without loss of generality we assume that $t > s$. Thanks to (2.21) of Remark 2.4 we have

$$\begin{aligned} \|\mathbf{u}(t) - \mathbf{u}(s)\|_\beta^2 &\leq C |A^{\beta-\frac{1}{4}}(I - e^{-(t-s)A})A^{\frac{1}{4}}\mathbf{u}(s)|^2 + C \left| \int_s^t A^\beta e^{-(t-r)A} B(\mathbf{u}(r), \mathbf{u}(r)) dr \right|^2 \\ &\quad + C \left| \int_s^t A^\beta e^{-(t-r)A} G(\mathbf{u}(r)) dW(r) \right|^2 \end{aligned}$$

Before proceeding further we recall that there exists a constant $C > 0$ such that for any $\gamma > 0$ and $t \geq 0$,

$$\|A^{-\gamma}(I - e^{-tA})\|_{\mathcal{L}(H)} \leq Ct^\gamma.$$

Applying this inequality, the Hölder inequality, Lemma 5.1-(B1), the Itô isometry and Assumption (G) imply

$$\begin{aligned} \mathbb{E} \left(\|\mathbf{u}(t) - \mathbf{u}(s)\|_\beta^2 \right) &\leq C(t-s) \mathbb{E} \left(\int_s^t (t-r)^{-2\beta} \|\mathbf{u}(r)\|_{\frac{1}{4}}^2 \|\mathbf{u}(r)\|_{\frac{1}{2}-\frac{1}{4}}^2 dr \right) \\ &\quad + C(t-s)^{2(\frac{1}{4}-\beta)} \mathbb{E} \|\mathbf{u}(s)\|_{\frac{1}{4}}^2 + \mathbb{E} \int_s^t |e^{-(t-r)A} A^\beta G(\mathbf{u}(r))|^2 dr \\ &\leq C(t-s)^{2-2\beta} \mathbb{E} \left(\sup_{r \in [s, t]} \|\mathbf{u}(r)\|_{\frac{1}{4}}^2 \sup_{r \in [s, t]} \|\mathbf{u}(r)\|_{\frac{1}{2}-\frac{1}{4}}^2 \right) \\ &\quad + C[(t-s)^{2(\frac{1}{4}-\beta)} + (t-s)] \mathbb{E} \left(\sup_{r \in [s, t]} \|\mathbf{u}(r)\|_{\frac{1}{4}}^4 \right), \end{aligned}$$

from which along with (2.13) we easily infer that

$$\mathbb{E} \left(\|\mathbf{u}(t) - \mathbf{u}(s)\|_\beta^2 \right) \leq C[(t-s)^{2-2\beta} + (t-s)^{2(\frac{1}{4}-\beta)} + (t-s)].$$

Thus, we have just finished the proof of the first part of the lemma.

Proof of item (ii). Let $t > s \geq 0$. Using (2.21) of Remark 2.4, it is not difficult to see that

$$\begin{aligned} \int_s^t \|\mathbf{u}(t) - \mathbf{u}(r)\|_{\frac{1}{2}+\beta}^2 dr &\leq C \int_s^t \left(\int_r^t |A^{\frac{1}{2}+\beta} e^{-(t-\tau)A} B(\mathbf{u}(\tau), \mathbf{u}(\tau))| d\tau \right)^2 dr \\ &\quad + C \int_s^t \left| \int_r^t A^{\frac{1}{4}+\beta} e^{-(t-\tau)A} [A^{\frac{1}{4}} G(\mathbf{u}(\tau))] dW(\tau) \right|^2 dr \\ &\quad + C \int_s^t |A^{\beta-\frac{1}{4}} (e^{-(t-r)A} - I) A^{\frac{3}{4}} \mathbf{u}(s)|^2 dr, \end{aligned}$$

from which and the assumption on B we infer that

$$\begin{aligned} \int_s^t \|\mathbf{u}(t) - \mathbf{u}(r)\|_{\frac{1}{2}+\beta}^2 dr &\leq C \sup_{0 \leq \tau \leq T} \left(\|\mathbf{u}(\tau)\|_{\frac{1}{4}}^2 \|\mathbf{u}(\tau)\|_{\frac{1}{2}-\frac{1}{4}}^2 \right) \int_s^t \left(\int_r^t (t-\tau)^{-\frac{1}{2}-\beta} d\tau \right)^2 dr \\ &\quad + C \int_s^t \left| \int_r^t A^{\frac{1}{4}+\beta} e^{-(t-\tau)A} [A^{\frac{1}{4}} G(\mathbf{u}(\tau))] dW(\tau) \right|^2 dr \\ &\quad + C \int_s^t (t-r)^{2(\frac{1}{4}-\beta)} \|\mathbf{u}(s)\|_{\frac{3}{4}}^2 dr. \end{aligned}$$

Taking the mathematical expectation and using (2.13)

$$\begin{aligned} \mathbb{E} \left(\mathbb{1}_{\Omega_k} \int_s^t \|\mathbf{u}(t) - \mathbf{u}(r)\|_{\frac{1}{2}+\beta}^2 dr \right) &\leq C(t-s)^{2-2\beta} + C(t-s)^{2(\frac{1}{4}-\beta)} \mathbb{E} \int_0^T \|\mathbf{u}(r)\|_{\frac{1}{2}+\beta}^2 dr \\ &\quad + \int_s^t \mathbb{E} \left(\left| \int_r^t A^{\frac{1}{4}+\beta} e^{-(t-\tau)A} [A^{\frac{1}{4}} G(\mathbf{u}(\tau))] dW(\tau) \right|^2 \right) dr. \end{aligned}$$

Owing Itô isometry, the assumption (G) and (2.13), we obtain

$$\begin{aligned} \mathbb{E} \left(\int_s^t \|\mathbf{u}(t) - \mathbf{u}(r)\|_{\frac{1}{2}+\beta}^2 dr \right) &\leq \mathbb{E} \left(\sup_{0 \leq \tau \leq T} (1 + \|\mathbf{u}(\tau)\|_{\frac{1}{4}}^2) \right) \int_s^t \int_r^t (t-\tau)^{-\frac{1}{2}-2\beta} d\tau dr \\ &\quad + (t-s)^{2-2\beta} + (t-s)^{2(\frac{1}{4}-\beta)}. \end{aligned}$$

Therefore, there exists a constant $C > 0$ such that

$$\mathbb{E} \left(\int_s^t \|\mathbf{u}(t) - \mathbf{u}(r)\|_{\frac{1}{2}+\beta}^2 dr \right) \leq C(t-s)^{2-2\beta} + C(t-s)^{2(\frac{1}{4}-\beta)} + C(t-s)^{\frac{3}{2}-2\beta},$$

for any $t > s \geq 0$. □

Proof of Theorem 2.6. Since the embedding $V_\beta \subset H$ is continuous for $\beta \in (0, \frac{1}{4})$, it is sufficient to prove the main theorem for $\beta \in (0, \frac{1}{4})$.

Note that the numerical scheme (2.24) is equivalent to

$$(U^{j+1}, w) + \int_{t_j}^{t_{j+1}} \langle AU^{j+1} + \pi_N B(U^j, U^{j+1}), w \rangle ds = (U^j, w) + \left(\int_{t_j}^{t_{j+1}} \pi_N G(U^j) dW(s), w \right), \quad (4.3)$$

for any $j \in \{1, \dots, M\}$ and $w \in V$. Integrating (1.1) and subtracting the resulting equation and the identity (4.3) term by term yields

$$\begin{aligned} (\mathbf{e}^{j+1} - \mathbf{e}^j, w) + \int_{t_j}^{t_{j+1}} \langle A\mathbf{e}^{j+1} + A(\mathbf{u}(s) - \mathbf{u}(t_{j+1})) + B(\mathbf{u}(s), \mathbf{u}(s)) - \pi_N B(U^j, U^{j+1}), w \rangle ds \\ = \left(\int_{t_j}^{t_{j+1}} [G(\mathbf{u}(s)) - \pi_N G(U^j)] dW(s), w \right). \end{aligned} \quad (4.4)$$

Observe that if $\mathbf{v} \in D(A^{\frac{1}{2}+\alpha})$ with $\alpha > \beta$, then $A^{2\beta}\mathbf{v} \in D(A^{\frac{1}{2}+\alpha-\beta}) \subset D(A^{\frac{1}{2}-\alpha})$, $A \in D(A^{\alpha-\frac{1}{2}})$ and the duality product $\langle A\mathbf{v}, A^{2\beta}\mathbf{v} \rangle$ is meaningful. Thus, we are permitted to take $w = 2A^{2\beta}\mathbf{e}^{j+1}$ in (4.4) and derive that

$$\begin{aligned} & \|\mathbf{e}^{j+1}\|_\beta^2 - \|\mathbf{e}^j\|_\beta^2 + \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2 + 2k\|\mathbf{e}^{j+1}\|_{\frac{1}{2}+\beta}^2 - 2 \int_{t_j}^{t_{j+1}} \|A^{\frac{1}{2}+\beta}(\mathbf{u}(s) - \mathbf{u}(t_{j+1}))\|_{\frac{1}{2}+\beta} \|\mathbf{e}^{j+1}\|_{\frac{1}{2}+\beta} ds \\ & \leq 2 \int_{t_j}^{t_{j+1}} \left| (A^{\beta-\frac{1}{2}}[B(\mathbf{u}(s), \mathbf{u}(s)) - \pi_N B(\mathbf{U}^j, \mathbf{U}^{j+1})], A^{\frac{1}{2}+\beta} \mathbf{e}^{j+1}) \right| ds \\ & \quad + 2 \left(\int_{t_j}^{t_{j+1}} [G(\mathbf{u}(s)) - \pi_N G(\mathbf{U}^j)] dW(s), A^{2\beta} \mathbf{e}^{j+1} \right), \end{aligned}$$

where we have used the identity $(\mathbf{v} - \mathbf{x}, 2A^{2\beta}\mathbf{v}) = \|\mathbf{v}\|_\beta^2 - \|\mathbf{x}\|_\beta^2 + \|\mathbf{v} - \mathbf{x}\|_\beta^2$. Now, by using the identity $\mathbf{v} = (\pi_N + [\mathbf{I} - \pi_N])\mathbf{v}$, the fact that

$$B(\mathbf{u}(s), \mathbf{u}(s)) - \pi_N B(\mathbf{U}^j, \mathbf{U}^{j+1}) = B(\mathbf{u}(s), \mathbf{u}(s)) - \pi_N B(\mathbf{u}(t_j), \mathbf{u}(t_{j+1})) + \pi_N B(\mathbf{u}(t_j), \mathbf{u}(t_{j+1})) - B(\mathbf{U}^j, \mathbf{U}^{j+1}),$$

the Cauchy-Schwarz inequality, the Cauchy inequality $ab \leq \frac{a^2}{4} + b^2$, $a, b > 0$ and the Assumption (B1) we obtain

$$\|\mathbf{e}^{j+1}\|_\beta^2 - \|\mathbf{e}^j\|_\beta^2 + \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2 + k\|\mathbf{e}^{j+1}\|_{\frac{1}{2}+\beta}^2 \leq 2\mathcal{L}_j + 16C_0^2 \sum_{i=1}^5 \mathcal{N}_{j,i} + 2\mathcal{W}_j, \quad (4.5)$$

where for each $j \in \{0, M-1\}$ the symbols \mathcal{L}_j , $\mathcal{N}_{j,i}$, $i = 1, \dots, 5$, and \mathcal{W}_j are defined by

$$\begin{aligned} \mathcal{L}_j &:= \int_{t_j}^{t_{j+1}} \|\mathbf{u}(s) - \mathbf{u}(t_{j+1})\|_{\frac{1}{2}+\beta}^2 ds, \\ \mathcal{N}_{j,1} &:= \int_{t_j}^{t_{j+1}} \|\mathbf{u}(s) - \mathbf{u}(t_{j+1})\|_\beta^2 (\|\mathbf{U}^j\|_\beta^2 + \|\mathbf{u}(s)\|_\beta^2) ds, \\ \mathcal{N}_{j,2} &:= \int_{t_j}^{t_{j+1}} \|\mathbf{e}^{j+1}\|_\beta^2 (\|\mathbf{U}^{j+1}\|_\beta^2 + \|\mathbf{u}(s)\|_\beta^2) ds, \\ \mathcal{N}_{j,3} &:= \int_{t_j}^{t_{j+1}} \|\mathbf{u}(s) - \mathbf{u}(t_j)\|_\beta^2 (|\mathbf{U}^{j+1}|^2 + |\mathbf{u}(s)|^2) ds, \\ \mathcal{N}_{j,4} &:= \int_{t_j}^{t_{j+1}} \|\mathbf{e}^j\|_\beta^2 (|\mathbf{U}^{j+1}|^2 + |\mathbf{u}(s)|^2) ds, \\ \mathcal{N}_{j,5} &:= \int_{t_j}^{t_{j+1}} \|(\mathbf{I} - \pi_N)B(\mathbf{u}(s), \mathbf{u}(s))\|_{\beta-\frac{1}{2}}^2 ds, \\ \mathcal{W}_j &:= \left(\int_{t_j}^{t_{j+1}} [G(\mathbf{u}(s)) - \pi_N G(\mathbf{U}^j)] dW(s), A^{2\beta} \mathbf{e}^{j+1} \right). \end{aligned}$$

Let $m \in [1, M]$ an arbitrary integer. Summing (4.5) from $j = 0$ to $m-1$ then taking the maximum over $m \in [0, M]$, multiplying by $\mathbb{1}_{\Omega_k}$ and taking the mathematical expectation imply

$$\begin{aligned} & \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^m\|_\beta^2 \right] + \sum_{j=0}^{M-1} \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2 \right] \\ & + k \sum_{j=1}^M \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^j\|_{\frac{1}{2}+\beta}^2 \right] \leq \mathbb{E} \|\mathbf{e}^0\|_\beta^2 + 16C_0^2 \sum_{j=0}^{M-1} \sum_{i=1}^5 \mathbb{E} [\mathbb{1}_{\Omega_k} \mathcal{N}_{j,i}] \\ & + 2 \sum_{j=0}^{M-1} \mathbb{E} [\mathbb{1}_{\Omega_k} \mathcal{L}_j] + 16C_0^2 \sum_{j=0}^{M-1} \mathcal{N}_{j,5} + 2 \sum_{j=0}^{m-1} \mathbb{E} [\mathbb{1}_{\Omega_k} \mathcal{W}_j]. \end{aligned}$$

Invoking the two items of Lemma 4.1 and the fact that $\|\mathbf{u}(s)\|_\beta^2 + \max_{0 \leq j \leq M} \|U^j\|_\beta^2 \leq f(k)$ on the set Ω_k we infer that

$$\begin{aligned} & \mathbb{E} \left[\mathbb{1}_{\Omega_k} \max_{0 \leq m \leq M} \|\mathbf{e}^m\|_\beta^2 \right] + \sum_{j=0}^{M-1} \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2 \right] + k \sum_{j=1}^M \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^j\|_{\frac{1}{2}+\beta}^2 \right] \\ & \leq \mathbb{E} \|\mathbf{e}^0\|_\beta^2 + 16C_0^2 k f(k) \sum_{m=0}^{M-1} \mathbb{E} \left(\mathbb{1}_{\Omega_k} [\|\mathbf{e}^{m+1}\|_\beta^2 + \|\mathbf{e}^m\|_\beta^2] \right) + 2C_8 f(k) M k [\Psi(k) + k^{1+\frac{1}{2}-\beta}] \\ & \quad + 64C_0^2 C_8 [f(k)]^2 M k [\Psi(k) + k] + 16C_0^2 \sum_{j=0}^{M-1} \mathcal{N}_{j,5} + 2 \sum_{j=0}^{M-1} \mathbb{E} [\mathbb{1}_{\Omega_k} \mathcal{W}_j], \end{aligned} \quad (4.6)$$

where $\psi(k) := k^{2-2\beta} + k^{2(\frac{1}{4}-\beta)}$. Now, thanks to Assumption (B1)' we have

$$\begin{aligned} \mathbb{1}_{\Omega_k} \int_{t_j}^{t_{j+1}} \|(I - \pi_N)B(\mathbf{u}(s), \mathbf{u}(s))\|_{\beta-\frac{1}{2}}^2 ds &= \mathbb{1}_{\Omega_k} \int_{t_j}^{t_{j+1}} \sum_{n=N+1}^{\infty} \lambda_n^{2\beta-1} |B_n(\mathbf{u}(s), \mathbf{u}(s))|^2 ds \\ &\leq \lambda_N^{2\beta-1} \int_{t_j}^{t_{j+1}} \mathbb{1}_{\Omega_k} \sum_{n=0}^{\infty} |B_n(\mathbf{u}(s), \mathbf{u}(s))|^2 ds \\ &\leq \lambda_N^{2\beta-1} \int_{t_j}^{t_{j+1}} \mathbb{1}_{\Omega_k} |B(\mathbf{u}(s), \mathbf{u}(s))|^2 ds \\ &\leq C \lambda_N^{2\beta-1} k \sup_{s \in [0, T]} \|\mathbf{u}(s)\|_{\frac{1}{4}}^4. \end{aligned}$$

Hence, owing to (2.13) we find a constant $C > 0$ such that

$$\mathbb{E} \mathbb{1}_{\Omega_k} \int_{t_j}^{t_{j+1}} \|(I - \pi_N)B(\mathbf{u}(s), \mathbf{u}(s))\|_{\beta-\frac{1}{2}}^2 ds \leq C \lambda_N^{2\beta-1} k.$$

Notice also that

$$\begin{aligned} & \sum_{j=0}^{M-1} \|\mathbf{e}^{j+1}\|_\beta^2 (\|U^{j+1}\|_\beta^2 + \|\mathbf{u}(s)\|_\beta^2) \\ &= \sum_{j=0}^{M-1} \|U^{j+1} - U^j + U^j - \mathbf{u}(t_j) + \mathbf{u}(t_j) - \mathbf{u}(t_{j+1})\|_\beta^2 (\|U^{j+1}\|_\beta^2 + \|\mathbf{u}(s)\|_\beta^2) \\ &\leq 3 \sum_{j=0}^{M-1} \left(\|U^{j+1} - U^j\|_\beta^2 + \|\mathbf{e}^j\|_\beta^2 + \|\mathbf{u}(t_j) - \mathbf{u}(t_{j+1})\|_\beta^2 \right) \left(\max_{0 \leq j \leq M} \|U^{j+1}\|_\beta^2 + \|\mathbf{u}(s)\|_\beta^2 \right). \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left(\mathbb{1}_{\Omega_k} \sum_{j=0}^{M-1} \|\mathbf{e}^{j+1}\|_\beta^2 (\|U^{j+1}\|_\beta^2 + \|\mathbf{u}(s)\|_\beta^2) \right) - C f(k) \mathbb{E} \sum_{m=0}^{M-1} \|\mathbf{e}^m\|_\beta^2 + f(k) C_7 [\psi(k) + k] \\ & \leq C \left(\mathbb{E} \left(\sum_{j=0}^{M-1} \|U^{j+1} - U^j\|_\beta^2 \right)^2 \right)^{\frac{1}{2}} \left(\mathbb{E} \max_{0 \leq j \leq M} \|U^j\|_\beta^4 + \mathbb{E} \sup_{s \in [0, T]} \|\mathbf{u}(s)\|_\beta^4 \right)^{\frac{1}{2}}. \end{aligned}$$

As long as the initial data is concerned, we have

$$\mathbb{E} \|\mathbf{e}^0\|_\beta^2 = \|[\pi_N + (I - \pi_N)]\mathbf{u}_0 - \pi_N \mathbf{u}_0\|_\beta^2 \quad (4.7)$$

$$\leq \sum_{n=N+1}^{\infty} \lambda_n^{2(\beta-\frac{1}{4})} \lambda_{\frac{1}{N}}^{\frac{1}{2}} |\mathbf{u}_{0,n}|^2 \quad (4.8)$$

$$\leq \lambda_N^{2(\beta-\frac{1}{4})} \|\mathbf{u}_0\|^2. \quad (4.9)$$

From all the above observations, (4.6), Assumption (B1)', (2.25)-(2.27) and (2.29) we infer that there exists a constant $C_9 > 0$ such that

$$\begin{aligned} & \mathbb{E} \left[\mathbb{1}_{\Omega_k} \max_{0 \leq m \leq M} \|\mathbf{e}^m\|_\beta^2 \right] + \sum_{j=0}^{M-1} \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2 \right] + k \sum_{j=1}^M \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^j\|_{\frac{1}{2}+\beta}^2 \right] \\ & \leq C_9 f(k) [\Psi(k) + k^{1+\frac{1}{2}-\beta}] + C_9 f(k) [\Psi(k) + k] + C_9 \left(\lambda_N^{2\beta-1} + \lambda^{2(\beta-\frac{1}{4})} \right) + 2 \sum_{j=0}^{M-1} \mathbb{E} [\mathbb{1}_{\Omega_k} \mathscr{W}_j]. \quad (4.10) \\ & \quad + C_9 k f(k) \sum_{m=0}^{M-1} \mathbb{E} \left[\mathbb{1}_{\Omega_k} \sup_{0 \leq j \leq m} \|\mathbf{e}^j\|_\beta^\beta \right] + 16 C_0^2 k f(k) \mathbb{E} \left[\mathbb{1}_{\Omega_k} \max_{0 \leq m \leq M} \|\mathbf{e}^m\|_\beta^2 \right]. \end{aligned}$$

Now we deal with the term containing \mathscr{W}_j . After subtracting from \mathscr{W}_j the martingale M_0 with mean zero defined by

$$M_0 = \left(A^\beta \int_{t_j}^{t_{j+1}} [G(\mathbf{u}(s)) - \pi_N G(U^j)] dW(s), A^\beta \mathbf{e}^j \right),$$

then taking the mathematical expectation, using the Young inequality and the Itô isometry give

$$\begin{aligned} \mathbb{E} \mathbb{1}_{\Omega_k} \mathscr{W}_j & \leq C \mathbb{E} \mathbb{1}_{\Omega_k} \left\| \int_{t_j}^{t_{j+1}} [G(\mathbf{u}(s)) - \pi_N G(U^j)] dW(s) \right\|_\beta^2 + \frac{1}{4} \mathbb{E} \mathbb{1}_{\Omega_k} \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2, \\ & \leq C \int_{t_j}^{t_{j+1}} \mathbb{E} \mathbb{1}_{\Omega_k} \|G(\mathbf{u}(s)) - \pi_N G(U^j)\|_{\mathcal{L}(\mathcal{H}, V_\beta)}^2 ds + \frac{1}{4} \mathbb{E} \mathbb{1}_{\Omega_k} \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2, \\ & \leq \sum_{i=1}^3 \mathbb{E} [\mathbb{1}_{\Omega_k} \mathscr{W}_{j,i}] + \frac{1}{4} \mathbb{E} \mathbb{1}_{\Omega_k} \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2, \end{aligned}$$

where the first two symbols $\mathscr{W}_{j,i}$, $i \in \{1, 2\}$ satisfy the following equalities and inequalities

$$\begin{aligned} \mathbb{E} [\mathbb{1}_{\Omega_k} \mathscr{W}_{j,1}] & = C \int_{t_j}^{t_{j+1}} \mathbb{E} \mathbb{1}_{\Omega_k} \|\pi_N G(\mathbf{u}(s)) - \pi_N G(\mathbf{u}(t_j))\|_{\mathcal{L}(\mathcal{H}, V_\beta)}^2 ds, \\ & \leq C C_3^2 \int_{t_j}^{t_{j+1}} \mathbb{E} \|\mathbf{u}(s) - \mathbf{u}(t_j)\|_\beta^2 ds, \\ & \leq C C_3^2 C_7^2 k [k^{2-2\beta} + k^{2(\frac{1}{4}-\beta)} + k]; \\ \mathbb{E} [\mathbb{1}_{\Omega_k} \mathscr{W}_{j,2}] & = C \int_{t_j}^{t_{j+1}} \mathbb{E} \mathbb{1}_{\Omega_k} \|\pi_N G(\mathbf{u}(t_j)) - \pi_N G(U^j)\|_{\mathcal{L}(\mathcal{H}, V_\beta)}^2 ds, \\ & \leq C C_3^2 k \mathbb{E} \mathbb{1}_{\Omega_k} \|\mathbf{e}^j\|_\beta^2, \end{aligned}$$

where Lemma 4.1 was used to get the last line.

The third term $\mathscr{W}_{j,3}$ satisfies

$$\begin{aligned} \mathbb{E} [\mathbb{1}_{\Omega_k} \mathscr{W}_{j,3}] & = \int_{t_j}^{t_{j+1}} \mathbb{E} \left(\mathbb{1}_{\Omega_k} \|(I - \pi_N)G(\mathbf{u}(s))\|_{\mathcal{L}(\mathcal{H}, V_\beta)}^2 \right) ds, \\ & = \int_{t_j}^{t_{j+1}} \mathbb{E} \left(\mathbb{1}_{\Omega_k} \sum_{n=N+1}^{\infty} \lambda_n^{2(\beta-\frac{1}{4})} \lambda_n^{\frac{1}{2}} \sup_{h \in \mathcal{H}, \|h\|_{\mathcal{H}} \leq 1} |G_n(\mathbf{u}(s))h|^2 \right) ds \\ & \leq \lambda_N^{2(\beta-\frac{1}{4})} \int_{t_j}^{t_{j+1}} \mathbb{E} \left(\mathbb{1}_{\Omega_k} \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \sup_{h \in \mathcal{H}, \|h\|_{\mathcal{H}} \leq 1} |G_n(\mathbf{u}(s))h|^2 \right) ds \\ & \leq \lambda_N^{2(\beta-\frac{1}{4})} k \mathbb{E} \left(\mathbb{1}_{\Omega_k} \sup_{s \in [0, T]} \|G(\mathbf{u}(s))\|_{\mathcal{L}(\mathcal{H}, V_{\frac{1}{4}})}^2 \right). \end{aligned}$$

Now, using Assumption (G) and the estimate (2.13) we infer that

$$\mathbb{E}[\mathbb{1}_{\Omega_k} \mathcal{W}_{j,3}] \leq CC_3^2 \lambda_N^{2(\beta-\frac{1}{4})} k,$$

for any $j \in [0, M]$. Thus, summing up we have obtained that

$$\begin{aligned} 2 \sum_{j=0}^{M-1} \mathbb{E}[\mathbb{1}_{\Omega_k} \mathcal{W}_j] &\leq CC_3^2 C_7^2 T[\psi(k) + k] + CC_3^2 T \lambda_N^{2(\beta-\frac{1}{4})} \\ &+ CC_3^2 k \sum_{m=0}^{M-1} \mathbb{E}[\mathbb{1}_{\Omega_k} \max_{0 \leq j \leq m} \|\mathbf{e}^j\|_\beta^2] + \frac{1}{2} \sum_{m=0}^{M-1} \mathbb{E}(\mathbb{1}_{\Omega_k} \|\mathbf{e}^{m+1} - \mathbf{e}^m\|_\beta^2). \end{aligned}$$

By plugging this last estimate into (4.6), we find a constant $C_{10} > 0$ such that

$$\begin{aligned} &\mathbb{E} \left[\mathbb{1}_{\Omega_k} \max_{0 \leq m \leq M} \|\mathbf{e}^m\|_\beta^2 \right] + \sum_{j=0}^{M-1} \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2 \right] + 2k \sum_{j=1}^M \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^j\|_{\frac{1}{2}+\beta}^2 \right] \\ &\leq C_{10} f(k) [\Psi(k) + k + k^{1+\frac{1}{2}-\beta}] + C_{10} f(k) [\Psi(k) + k] + C_{10} \lambda_N^{2\beta-1} + C_{10} \lambda_N^{2(\beta-\frac{1}{4})} \\ &\quad + C_{10} k [f(k) + 1] \sum_{m=0}^{M-1} \mathbb{E} \left[\mathbb{1}_{\Omega_k} \sup_{0 \leq j \leq m} \|\mathbf{e}^j\|_\beta^\beta \right]. \end{aligned}$$

Now, an application of the discrete Gronwall lemma yields

$$\begin{aligned} &\mathbb{E} \left[\mathbb{1}_{\Omega_k} \max_{0 \leq m \leq M} \|\mathbf{e}^m\|_\beta^2 \right] + \sum_{j=0}^{M-1} \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2 \right] + 2k \sum_{j=1}^M \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^j\|_{\frac{1}{2}+\beta}^2 \right] \\ &\leq \left(C_{10} f(k) [\Psi(k) + k + k^{1+\frac{1}{2}-\beta}] + C_{10} f(k) [\Psi(k) + k] + C_{10} \lambda_N^{2\beta-1} + C_{10} \lambda_N^{2(\beta-\frac{1}{4})} \right) e^{C_{10} T [f(k)+1]}. \end{aligned}$$

Since

$$\min\{k^{2-2\beta}, k^{1+\frac{1}{2}-\beta}, k^{2(\frac{1}{4}-\beta)}, k\} = k^{2(\frac{1}{4}-\beta)} \text{ and } \min\{\lambda_N^{2(\beta-\frac{1}{4})}, \lambda_N^{2\beta-1}\} = \lambda_N^{2(\beta-\frac{1}{4})},$$

for any $\beta \in [0, \frac{1}{4}]$, and $k^\varepsilon f(k) = k^\varepsilon \log k^{-\varepsilon} \leq \frac{1}{2}$ for any $k > 0$ and $\varepsilon \in (0, 2(\frac{1}{4} - \beta))$, we derive that there exists a constant $C > 0$ such that

$$\begin{aligned} &\mathbb{E} \left[\mathbb{1}_{\Omega_k} \max_{0 \leq m \leq M} \|\mathbf{e}^m\|_\beta^2 \right] + \sum_{j=0}^{M-1} \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^{j+1} - \mathbf{e}^j\|_\beta^2 \right] \\ &+ 2k \sum_{j=1}^M \mathbb{E} \left[\mathbb{1}_{\Omega_k} \|\mathbf{e}^j\|_{\frac{1}{2}+\beta}^2 \right] \leq C k^{-2\varepsilon} [k^{2(\frac{1}{4}-\beta)} + \lambda_N^{-2(\frac{1}{4}-\beta)}]. \end{aligned} \tag{4.11}$$

This estimate completes the proof of the Theorem 2.6. \square

5. Motivating Examples

In this section we give two examples of evolution equations to which we can apply our abstract result.

5.1. Stochastic GOY and Sabra shell models

The first examples we can take is the GOY and Sabra shell models. To describe this model let us denote by \mathbb{C} the field of complex numbers, $\mathbb{C}^{\mathbb{N}}$ the set of all \mathbb{C} -valued sequences, and we set

$$\mathbf{H} = \left\{ \mathbf{u} = (\mathbf{u}_n)_{n \in \mathbb{N}} \subset \mathbb{C}; \sum_{n=1}^{\infty} |\mathbf{u}_n|^2 < \infty \right\}.$$

let k_0 be a positive number and $\lambda_n = k_0 2^n$ be a sequence of positive numbers. The space H is a separable Hilbert space when endowed with the scalar product defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^{\infty} \mathbf{u}_k \bar{\mathbf{v}}_k, \text{ for } \mathbf{u}, \mathbf{v} \in H,$$

where \bar{z} denotes the conjugate of any complex number z .

We define a linear map A with domain

$$D(A) = \{ \mathbf{u} \in H; \sum_{n=1}^{\infty} \lambda_n^4 |\mathbf{u}_n|^2 < \infty \},$$

by setting

$$A\mathbf{u} = (\lambda_n^2 \mathbf{u}_n)_{n \in \mathbb{N}}, \text{ for } \mathbf{u} \in D(A).$$

It is not hard to check that A is a self-adjoint and strictly positive operator. Moreover, the embedding $D(A^\alpha) \subset D(A^{\alpha+\varepsilon})$ is compact for any $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. The spaces $D(A^\alpha)$, $\alpha \in \mathbb{R}$, can be characterized as follow

$$D(A^\alpha) = \{ \mathbf{u} = (\mathbf{u}_n)_{n \in \mathbb{N}} \subset \mathbb{C}; \sum_{n=1}^{\infty} \lambda_n^{4\alpha} |\mathbf{u}_n|^2 < \infty \}.$$

For any $\alpha \in \mathbb{R}$ the space $V_\alpha = D(A^\alpha)$ is a separable Hilbert space when equipped with the scalar product

$$((\mathbf{u}, \mathbf{v}))_\alpha = \sum_{k=1}^{\infty} \lambda_k^{4\alpha} \mathbf{u}_k \bar{\mathbf{v}}_k, \text{ for } \mathbf{u}, \mathbf{v} \in V_\alpha. \quad (5.1)$$

The norm associated to this scalar product will be denoted by $\|\mathbf{u}\|_\alpha$, $\mathbf{u} \in V_\alpha$. In what follows we set $V = D(A^{\frac{1}{2}})$.

The evolution equation describing the GOY and Sabra shell models is given by

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= \kappa A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) + F, \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{aligned} \quad (5.2)$$

where $F = (F_n)_{n \in \mathbb{N}} \subset \mathbb{C}$ is an external perturbation and $B(\cdot, \cdot)$ is a bilinear map defined on $V \times V$ taking values in the dual space V^* . More precisely, we assume that the nonlinear term

$$\begin{aligned} B : \mathbb{C}^{\mathbb{N}} \times \mathbb{C}^{\mathbb{N}} &\rightarrow \mathbb{C}^{\mathbb{N}}, \\ (\mathbf{u}, \mathbf{v}) &\mapsto B(\mathbf{u}, \mathbf{v}) = (b_1(\mathbf{u}, \mathbf{v}), \dots, b_n(\mathbf{u}, \mathbf{v}), \dots) \end{aligned}$$

for the GOY shell model (see [17], [18]) is defined by

$$\begin{aligned} b_n(\mathbf{u}, \mathbf{v}) &:= (B(\mathbf{u}, \mathbf{v}))_n \\ &:= i\lambda_n \left(\frac{1}{4} \bar{v}_{n-1} \bar{u}_{n+1} - \frac{1}{2} (\bar{u}_{n+1} \bar{v}_{n+2} + \bar{v}_{n+1} \bar{u}_{n+2}) + \frac{1}{8} \bar{u}_{n-1} \bar{v}_{n-2} \right), \end{aligned}$$

and for the Sabra shell model (see [19], [3]) it is defined by

$$\begin{aligned} b_n(\mathbf{u}, \mathbf{v}) &:= (B(\mathbf{u}, \mathbf{v}))_n := \frac{i}{3} \lambda_{n+1} [\bar{v}_{n+1} u_{n+2} + 2 \bar{u}_{n+1} v_{n+2}] \\ &\quad + \frac{i}{3} \lambda_n [\bar{u}_{n-1} v_{n+1} - \bar{v}_{n-1} u_{n+1}] \\ &\quad + \frac{i}{3} \lambda_{n-1} [2 u_{n-1} v_{n-2} + u_{n-2} v_{n-1}], \end{aligned}$$

for any $\mathbf{u} = (u_1, \dots, u_n, \dots) \in \mathbb{C}^{\mathbb{N}}$ and $\mathbf{v} = (v_1, \dots, v_n, \dots) \in \mathbb{C}^{\mathbb{N}}$.

We will show in the next lemma that the nonlinear term $B(\cdot, \cdot)$ for the GOY and Sabra shell models satisfy the Assumptions (B1) to (B3). Since $B(\cdot, \cdot)$ is bilinear it is sufficient to prove (B1)'. In particular, it is sufficient to prove the following lemma.

Lemma 5.1. (a) For any non-negative numbers α and β such that $\alpha + \beta \in (0, \frac{1}{2}]$, there exists a constant $c_0 > 0$ such that

$$\|\mathbf{B}(\mathbf{u}, \mathbf{v})\|_{-\alpha} \leq c_0 \begin{cases} \|\mathbf{u}\|_{\frac{1}{2}-(\alpha+\beta)} \|\mathbf{v}\|_{\beta} & \text{for any } \mathbf{u} \in V_{\frac{1}{2}-(\alpha+\beta)}, \mathbf{v} \in V_{\beta} \\ \|\mathbf{u}\|_{\beta} \|\mathbf{v}\|_{\frac{1}{2}-(\alpha+\beta)} & \text{for any } \mathbf{v} \in V_{\frac{1}{2}-(\alpha+\beta)}, \mathbf{u} \in V_{\beta}. \end{cases} \quad (5.3)$$

(b) For any $\mathbf{u} \in H, \mathbf{v} \in V$

$$\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{v} \rangle = 0. \quad (5.4)$$

Proof. The item (b) was proved in [20, Proposition 1], thus we omit its proof.

Item (a) can be viewed as a generalization of [20, Proposition 1]. We will just prove the latter item for the Sabra shell model since the proofs for the two models are very similar. Let $\mathbf{u} \in V_{\frac{1}{2}-(\alpha+\beta)}, \mathbf{v} \in V_{\beta}$, and $\mathbf{w} \in V_{\alpha}$ such that $\|\mathbf{w}\|_{\alpha} \leq 1$. We have

$$\begin{aligned} |\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle| &= \left| \sum_{n=1}^{\infty} b_n(\mathbf{u}, \mathbf{v}) \bar{\mathbf{w}}_n \right| \leq \sum_{n=1}^{\infty} |b_n(\mathbf{u}, \mathbf{v})| |\mathbf{w}_n| \\ &\leq \frac{1}{3} \sum_{n=1}^{\infty} \lambda_{n+1} (|\mathbf{u}_{n+1}| \cdot |\mathbf{v}_{n+2}| + |\mathbf{u}_{n+2}| \cdot |\mathbf{v}_{n+1}|) |\mathbf{w}_n| \\ &\quad + \frac{1}{3} \sum_{n=1}^{\infty} \lambda_n (|\mathbf{u}_{n-1}| \cdot |\mathbf{v}_{n+1}| + |\mathbf{u}_{n+1}| \cdot |\mathbf{v}_{n-1}|) |\mathbf{w}_n| \\ &\quad + \frac{1}{3} \sum_{n=1}^{\infty} \lambda_{n-1} (|\mathbf{u}_{n-1}| \cdot |\mathbf{v}_{n-2}| + |\mathbf{u}_{n-2}| \cdot |\mathbf{v}_{n-1}|) |\mathbf{w}_n| \\ &\leq I_1 + I_2 + I_3. \end{aligned}$$

For the term I_1 we have

$$\begin{aligned} I_1 &\leq \frac{1}{3} \sum_{n=1}^{\infty} \lambda_{n+1} |\mathbf{u}_{n+1}| \cdot |\mathbf{v}_{n+2}| |\mathbf{w}_n| + \frac{1}{3} \sum_{n=1}^{\infty} \lambda_{n+1} |\mathbf{u}_{n+2}| \cdot |\mathbf{v}_{n+1}| |\mathbf{w}_n| \\ &\leq I_{1,1} + I_{1,2}. \end{aligned}$$

We will treat the term $I_{1,1}$. By Hölder's inequality we have

$$\begin{aligned} I_{1,1} &\leq \frac{1}{3} \sum_{n=1}^{\infty} k_0 2 \lambda^{1-2\alpha} |\mathbf{u}_{n+1}| \cdot |\mathbf{v}_{n+2}| \lambda_n^{2\alpha} |\mathbf{w}_n| \\ &\leq \frac{2}{3} k_0 \left(\sum_{n=1}^{\infty} k_0 2 \lambda_n^{2-4(\alpha+\beta)} |\mathbf{u}_{n+1}|^2 \lambda_n^{4\beta} |\mathbf{v}_{n+2}|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \lambda_n^{4\alpha} |\mathbf{w}_n|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Since $\|\mathbf{w}\|_{\alpha} \leq 1$ and $\lambda_{n+p} = k_0^p 2^p \lambda_n$ we can find a constant $C > 0$ depending only on α, β and k_0 such that

$$\begin{aligned} I_{1,1} &\leq C \left(\max_{k \in \mathbb{N}} \lambda_{n+1}^{2-4(\alpha+\beta)} |\mathbf{u}_{n+1}|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \lambda_{n+2}^{4\beta} |\mathbf{v}_n|^2 \right)^{\frac{1}{2}}, \\ &\leq C \left(\sum_{n=1}^{\frac{1}{2}} \lambda_{n+1}^{4[\frac{1}{2}-(\alpha+\beta)]} |\mathbf{u}_{n+1}|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \lambda_{n+2}^{4\beta} |\mathbf{v}_n|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

from which we easily derive that

$$I_{1,1} \leq C \|\mathbf{u}\|_{\frac{1}{2}-(\alpha+\beta)} \|\mathbf{v}\|_{\beta}.$$

One can use an analogous argument to show that

$$I_{1,2} \leq C \|\mathbf{u}\|_{\frac{1}{2}-(\alpha+\beta)} \|\mathbf{v}\|_{\beta}.$$

Hence,

$$I_1 \leq C \|\mathbf{u}\|_{\frac{1}{2}-(\alpha+\beta)} \|\mathbf{v}\|_{\beta}.$$

Using a similar argument we can also prove that for any non-negative numbers α and β satisfying $\alpha + \beta \in (0, \frac{1}{2}]$ there exists a constant $C > 0$ such that

$$I_2 + I_3 \leq C \|\mathbf{u}\|_{\frac{1}{2}-(\alpha+\beta)} \|\mathbf{v}\|_{\beta},$$

for any $\mathbf{u} \in V_{\frac{1}{2}-(\alpha+\beta)}$ and $\mathbf{v} \in V_{\beta}$. Therefore, for any any non-negative numbers α and β satisfying $\alpha + \beta \in (0, \frac{1}{2}]$ we can find a constant $C > 0$ such that

$$\|B(\mathbf{u}, \mathbf{v})\|_{-\alpha} \leq C \|\mathbf{u}\|_{\frac{1}{2}-(\alpha+\beta)} \|\mathbf{v}\|_{\beta},$$

for any $\mathbf{u} \in V_{\frac{1}{2}-(\alpha+\beta)}$ and $\mathbf{v} \in V_{\beta}$. Interchanging the role of \mathbf{u} and \mathbf{v} we obtain that for any two numbers α and β as above there exists a positive constant C such that

$$\|B(\mathbf{u}, \mathbf{v})\|_{-\alpha} \leq C \|\mathbf{v}\|_{\frac{1}{2}-(\alpha+\beta)} \|\mathbf{u}\|_{\beta},$$

for any $\mathbf{v} \in V_{\frac{1}{2}-(\alpha+\beta)}$ and $\mathbf{u} \in V_{\beta}$. Thus we have just completed the proof of the lemma for the Sabra shell model. As we mentioned earlier, the case of the GOY model can be dealt with a similar argument. \square

For more mathematical results related to shell models we refer to [21], [22], [23], [20] and references therein.

5.2. Stochastic nonlinear heat equation

Let \mathcal{O} be a bounded domain of \mathbb{R}^d , $d = 1, 2$. We assume that its boundary $\partial\mathcal{O}$ is of class \mathcal{C}^∞ . The second example we can treat is the stochastic version of the following nonlinear heat equation

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + |\mathbf{u}| \mathbf{u} = F, \quad (5.5a)$$

$$\mathbf{u} = 0 \text{ on } \partial\mathcal{O}, \quad (5.5b)$$

$$\mathbf{u}(0, x) = \mathbf{u}_0 \quad x \in \mathcal{O}. \quad (5.5c)$$

Throughout this section we will denote by $H^\theta(\mathcal{O})$, $\theta \in \mathbb{R}$, the (fractional) Sobolev spaces as defined in [10] and $H_0^1(\mathcal{O})$ be the space of functions $\mathbf{u} \in H^1$ such that $\mathbf{u}|_{\partial\mathcal{O}} = 0$. In particular, we set $H = L^2(\mathcal{O})$ and we denote its scalar product by (\cdot, \cdot) .

To set (5.5) into the abstract form required for the treatment of (1.1) we define a continuous bilinear map $\alpha : H_0^1(\mathcal{O}) \times H_0^1(\mathcal{O}) \rightarrow \mathbb{R}$ by setting

$$\alpha(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}),$$

for any $\mathbf{u}, \mathbf{v} \in H_0^1(\mathcal{O})$. Thanks to the Riesz representation there exists a densely linear map A with domain $D(A) \subset H$ such that

$$\langle A\mathbf{v}, \mathbf{u} \rangle = \alpha(\mathbf{v}, \mathbf{u}),$$

for any $\mathbf{u}, \mathbf{v} \in H_0^1(\mathcal{O})$. It is well known that A is a self-adjoint and definite positive and its eigenfunctions $\{\psi_n; n \in \mathbb{N}\} \subset \mathcal{C}^\infty(\mathcal{O})$ form an orthonormal basis of H . For any $\alpha \in \mathbb{R}$ we set $V_\alpha = D(A^\alpha)$, in particular we put $V := D(A^{\frac{1}{2}})$. Now, we consider a nonlinear map B defined on $H \times D(A^{\frac{1}{2}})$ or $D(A^{\frac{1}{2}}) \times H$ and taking values in $D(A^{-\frac{1}{2}})$ by setting

$$B(\mathbf{u}, \mathbf{v}) = |\mathbf{u}| \mathbf{v},$$

for any $(\mathbf{u}, \mathbf{v}) \in H \times D(A^{\frac{1}{2}})$ or $(\mathbf{u}, \mathbf{v}) \in D(A^{\frac{1}{2}}) \times H$. It is clear that

$$\langle A\mathbf{v} + B(\mathbf{u}, \mathbf{v}), \mathbf{v} \rangle \geq \|\mathbf{v}\|_{\frac{1}{2}}^2, \quad (5.6)$$

for any $\mathbf{u}, \mathbf{v} \in V$. Here we should note that thanks to the solution of Kato's square root problem in [24, Theorem 1] we have $\|\mathbf{u}\|_{\frac{1}{2}} \simeq |\nabla \mathbf{u}|$ for any $\mathbf{u} \in H_0^1(\mathcal{O})$, i.e, $V = H_0^1(\mathcal{O})$.

Now we claim that for any numbers $\alpha \in [0, \frac{1}{2})$ and $\beta \in (0, \frac{1}{2})$ such that $\alpha + \beta \in (0, \frac{1}{2})$, there exists a constant $c_0 > 0$ such that

$$\|B(\mathbf{u}, \mathbf{v})\|_{-\alpha} \leq c_0 \begin{cases} \|\mathbf{u}\|_{\frac{1}{2}-(\alpha+\beta)} \|\mathbf{v}\|_{\beta} & \text{for any } \mathbf{u} \in V_{\frac{1}{2}-(\alpha+\beta)}, \mathbf{v} \in V_{\beta} \\ \|\mathbf{u}\|_{\beta} \|\mathbf{v}\|_{\frac{1}{2}-(\alpha+\beta)} & \text{for any } \mathbf{v} \in V_{\frac{1}{2}-(\alpha+\beta)}, \mathbf{u} \in V_{\beta}, \end{cases} \quad (5.7)$$

and

$$\|B(\mathbf{u}, \mathbf{v})\|_{-\frac{1}{2}} \leq c_0 \|\mathbf{u}\|_{\frac{1}{4}} \|\mathbf{v}\|_{\frac{1}{4}} \quad \text{for any } \mathbf{v} \in V_{\frac{1}{4}}, \mathbf{u} \in V_{\frac{1}{4}}. \quad (5.8)$$

To prove these inequalities, let $\beta > 0$ such that $\alpha + \beta < \frac{1}{2}$. Since

$$\left(\frac{1}{2} - \alpha\right) + \left(\frac{1}{2} - 1 + 2(\alpha + \beta)\right) + \left(\frac{1}{2} - \beta\right) = 1,$$

we have

$$|\langle \mathbf{u} | \mathbf{v}, \mathbf{w} \rangle| \leq C_0 \|\mathbf{u}\|_{L^r} \|\mathbf{v}\|_{L^s} \|\mathbf{w}\|_{L^q}, \quad (5.9)$$

where the constants q, r, s is defined through

$$\frac{1}{q} = \frac{1}{2} - \alpha, \quad \frac{1}{s} = \alpha + \beta, \quad \frac{1}{r} = \frac{1}{2} - \beta.$$

Recall that $V_{\alpha} \subset H^{2\alpha} \subset L^q$ with $\frac{1}{q} = \frac{1}{2} - \alpha$ if $\alpha \in (0, \frac{1}{2})$ and $q \in [2, \infty)$ arbitrary if $\alpha = \frac{1}{2}$. Then, we derive from (5.9) that the second inequality in (5.7) holds. By interchanging the role of r and s we derive that the first inequality in (5.7) also holds. One can establish (5.8) with the same argument. The estimates (5.7) and (5.8) easily implies (2.2) and (2.6).

Now we need to check that $B(\cdot, \cdot)$ satisfies (2.3). For this purpose we observe that there exists a constant $C > 0$ such that

$$|B(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}\| \|\mathbf{v}\|_{L^\infty},$$

which with the continuous embedding $V_{\frac{1}{2}+\varepsilon} \subset L^\infty$ for any $\varepsilon > 0$ implies (2.3).

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